Reverse engineering of algebraic inequalities for system reliability predictions and enhancing processes in engineering

Michael Todinov
Oxford Brookes University, Oxford, UK
School of Engineering, Computing and Mathematics
mtodinov@brookes.ac.uk

Abstract—The paper examines the profound impact on the forecasted system reliability when one assumes average reliabilities on demand for components of various kinds but of the same type. In this paper, we use reverse engineering of a novel algebraic inequality to demonstrate that the prevalent practice of using average reliability on demand for components of the same type but different varieties to calculate system reliability on demand is fundamentally flawed. This approach can introduce significant errors due to the innate variability of components within a given type.

Additionally, the paper illustrates the optimization of engineering processes using reverse engineering of sub-additive algebraic inequalities based on concave power laws. Employing reverse engineering on these sub-additive inequalities has paved the way for strategies that enhance the performance of diverse industrial processes. The primary advantage of these sub-additive inequalities lies in their simplicity, rendering them particularly suitable for reverse engineering.

Keywords: system reliability, optimisation; reverse engineering; algebraic inequalities; concave functions

1. Introduction

Algebraic inequalities play a vital role in engineering, as they provide a means to describe and analyse constraints, tolerances, and optimization problems in various fields.

There is extensive literature on algebraic inequalities (Fink, 2000; Stelle,2004; Hardi et al,1999), methods of proof (Sedrakian and Sedrakian,2010; Su and Xiong, 2016). The use of algebraic inequalities in engineering (Lewis,1996; Ebeling, 1997; Childs, 2014; Cloud et al. 1998; Samuel and Weir, 1999) has been confined to determining upper and lower bounds, errors estimates and defining design constraints.

The usefulness of algebraic inequalities, however, is by no means limited to determining bounds and describing design constraints. Algebraic inequalities can also be physically interpreted and connected to real systems and processes.

The reverse engineering of algebraic inequalities, grounded in their physical interpretation, *offers an alternate method for probing physical reality*. Key applications of reverse engineering in algebraic inequalities were recently explored in (Todinov, 2023).

The steps involved in the reverse engineering of algebraic inequalities are given in Fig.1. These steps demonstrate that the reverse engineering of a valid algebraic inequality effectively projects a new property of the physical reality whose footprint is the algebraic inequality itself.

An important class of algebraic inequalities, which are particularly suited for reverse engineering, consists of those based on sub-additive functions.

This paper demonstrates that for numerous industrial processes, the reverse engineering of sub-additive algebraic inequalities directly paves the way for uncomplicated yet highly efficient optimization strategies.

Despite their indisputable usefulness in optimization, there seems to be a significant lack of discussion concerning the physical interpretation of sub-additive inequalities.

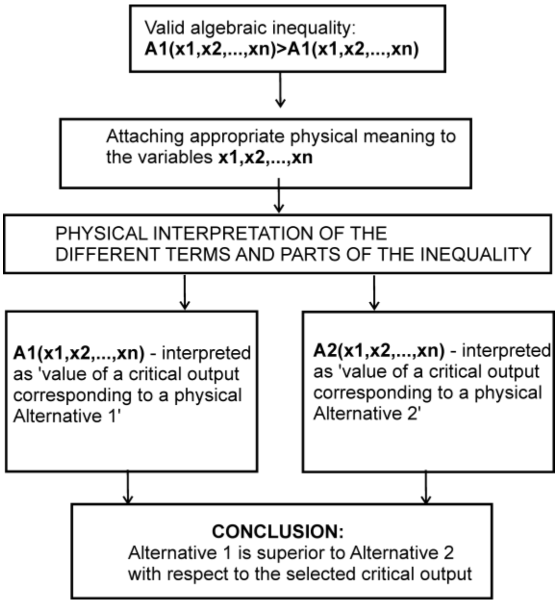

Figure 1. Key steps of the reverse engineering of a valid algebraic inequality

Concavity properties in reliability theory for continuous and discrete random variables have been discussed in (Alimohammadi et al, 2016). Sub-additive algebraic inequalities can be easily obtained from concave power law functions. The absence of physical interpretation of sub-additive inequalities based on concave power laws is particularly surprising given the prevalence of concave power-law dependencies in describing various processes in engineering.

This paper continues the theme related to reverse engineering of algebraic inequalities in two key directions:

- Using reverse engineering of algebraic inequalities to assess system reliability predictions based on average component reliabilities on demand and average hazard rates.

- Using reverse engineering of sub-additive inequalities based on concave power laws for optimising processes in engineering. These are also the main contributions of this paper.

2. System reliability predictions based on average component reliabilities

2.1 A general inequality related to series-parallel systems

Comparisons of systems with components logically arranged in series with distinct reliability functions and with the same, average reliability function have been made in (Navaro and Spizzichino, 2010).

In this section, we use reverse engineering of a key algebraic inequality to demonstrate *that the prevalent practice of using average reliability on demand for components of the same type but different varieties to calculate system reliability on demand is fundamentally flawed.*

Consider the next valid algebraic inequality:

$$(1-x_1^m)(1-x_2^m)...(1-x_n^m) \leq (1-[(x_1+x_2+...+x_n)/n]^m)^n \quad (1)$$

where $m \geq 1$ is an integer exponent and $x_1,...,x_n$ are n real values for which $0 \leq x_i \leq 1$. To the best of our knowledge, inequality (1) has never been reported before hence, a proof of this inequality has been provided in the Appendix.

2.2. Series-parallel systems

For $m = 2$, the general inequality (1) becomes

$$(1-x_1^2)(1-x_2^2)...(1-x_n^2) \leq (1-[(x_1+x_2+...+x_n)/n]^2)^n \quad (2)$$

Inequality (2) can be reverse-engineered easily. Let x_i ($0 < x_i < 1$) be physically interpreted as the probability of failure of a component C_i of variety *i*, where $i = 1,2,...,n$. Although all components are of the same type, they are inhomogeneous due to variations in properties, age and operating conditions.

The variables x_i are interpreted as *probability of failure on demand for components of variety i (all components are of the same type).* Note that the time is not present in the probability of failure on demand. Also, the probabilities of failure on demand for components of different varieties but of the same type *are not known* in advance. Therefore, the use of average values for the probability of failure on demand in reliability predictions, for these components, *is inevitable*.

Because of differences in age, the number and size of material and manufacturing flaws, and differences in working conditions, no two components of the same type are identical in terms of reliability. There is no way of knowing the reliability of a component of type *X* from a particular variety (with a specified age, number, nature and size of the material flaws and manufacturing flaws, working conditions, etc.).

For example, the reliability variation of components is significantly influenced by the presence of material flaws, as well as their size, number density, and location (Todinov, 2002, 2006). As a result, components of the same type and material, sourced from different suppliers, may exhibit considerable differences in their reliabilities. Since it is impossible to obtain the reliability on demand for components from different varieties, this intrinsic variability requires the use of average component reliability on demand (or probability of failure on demand). The average reliability on demand is what is listed in databases related to reliability on demand of components from a particular type. If, for example, 261 out of 900 valves of type *X* failed to close on command, the probability of failure on demand for valves of type *X* will be assessed by the average value of $261/900 = 0.29$.

An example of a system whose reliability depends on the probability of failure on demand of its components is given in Figure 2. The system consists of *n* pipelines transporting toxic fluid with two valves on each pipeline. All valves are initially open and a signal to close is sent to all valves in order to stop the fluid in each pipeline.

The system is deemed operational when, upon receiving a command for closure, the flow is halted in all *n* pipelines. To boost system reliability, each pipeline features a redundant valve. This redundancy implies that at least one valve on each pipeline must respond to the closure command to ensure that the flow in the pipeline is halted.

The valves are of varieties C1,C2,...,C*n*, characterised by probabilities of failure to close on demand $x_1, x_2, ..., x_n$, correspondingly.

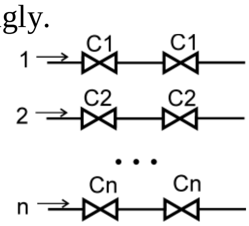

Figure 2. A system of *n* pipelines transporting toxic fluid with two valves of types C1,C2,...,Cn, on each pipeline.

Consider the reliability network of the system in Fig.2 which is given in Fig.3a.

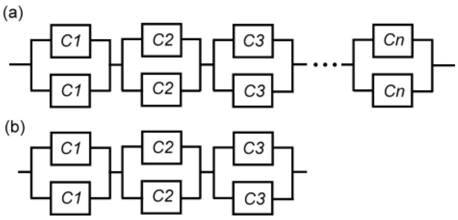

Figure 3. a) Reliability network of a series-parallel system with components from *n* varieties; b) Reliability network of a series-parallel system involving components of 3 varieties

It is a series-parallel system which is quite common in numerous engineering applications. For example, an alternative to the system illustrated in Fig.2, could be a system of *n* pipelines carrying toxic fluid, each equipped with a flange sealed by two seals one of which is redundant. Alternatively, in place of pipelines with flanges, we could consider a system consisting of *n* zones. In each zone, two sensors (one of which is redundant) measure the temperature, pressure, or concentration levels.

The number of varieties of components of type *X* will be denoted by *n*. The left-hand part of inequality (2) is the actual reliability of the system in Fig.3a. The expression $\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}$ in the right-hand part of (2) is the average probability of failure $\bar{x}$ on demand for the varieties of the selected type *X* (e.g. valve), assessed as an average related to *n* varieties.

Please note that the probabilities of failure x_i characterising the *n* varieties are not known and this is why the system reliability on demand cannot be estimated directly, by using these probabilities. Because the expression for $\bar{x}$ cannot be evaluated using x_i, the ratio p_f / p is always used instead where p_f is the number of observed in the past failed components from that type and *p* is the total number of observed components from that type. Note that p_f and p are obtained from component failure statistics and *are not related* to the number of components building the system.

The numbers of p_f and p should be sufficiently large to produce an accurate estimate of the probability of failure on demand p_f / p.

Thus, for a system with *n* components of the same type *X* and of different variety (*n* varieties in total), for the average probability of failure $\bar{x}$ of components in the system, from that particular type *X*, the following equation holds:

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} \approx \frac{p_f}{p} \qquad (3)$$

Equation (3) can be proved by considering that the left-hand side of (3) can be presented as

$$\frac{x_1 + x_2 + ... + x_n}{n} = (1/n) \times x_1 + (1/n) \times x_2 + ... + (1/n) x_n \qquad (4)$$

This essentially represents the total probability associated with the failure of a component in the system. Indeed, a component from type *X*, can fail in *n* mutually exclusive ways. This includes the scenario where the component belongs to variety 1 and fails (a compound event with probability $(1/n)x_1$), the scenario where the component belongs to variety 2 and fails (a compound event with probability $(1/n)x_2$), and so on.

The probability of failure of a component in the system must approach p_f / p because this ratio is the empirical probability of failure for a component of type *X*.

Very similar reasoning also applies to the case where the number *n* of component varieties is smaller than the number n_c of components in the system ($n < n_c$). Indeed, let $n_1, n_2, ..., n_n$ ($\sum_{i=1}^{n} n_i = n_c$) be the number of components in the system from each variety (these numbers are also unknown). The total probability of failure for a component in the system is then given by

$$\frac{p_f}{p} \approx \frac{n_1 x_1 + n_2 x_2 + ... + n_n x_n}{n_c} \qquad (5)$$

where p_f is the observed in the past total number of failed components (from failure statistics) and *p* is the total number of observed components in the past. The right-hand side of (5) is the weighted average of the probabilities of failure characterising the *n* varieties.

Indeed, a component in the system can fail in *n* mutually exclusive ways. This includes the scenario where the component belongs to variety 1 and fails (a compound event with probability $(n_1 / n_c) x_1$, the scenario where the component belongs to variety 2 and fails (a compound event with probability $(n_2 / n_c) x_2$), and so on.

The total probability of a component failure is then given by the expression:

$$(n_1 / n_c) x_1 + ... + (n_n / n_c) x_n$$

which must approach the empirical probability p_f / p of component failure and this yields equation (5).

To test equations (3) and (5), Monte Carlo simulations were performed, based on *p*=100000 observed components and *n*=1,2,...,10, component varieties. In an array, random values between 0 and 1 are initially assigned for the probabilities of failure characterising the *n* varieties. Next, *p*=100000 components were selected by choosing randomly their variety. Each randomly selected component was also virtually tested for failure on demand by using the probability of failure on demand characterising its variety. At the end of the simulation, the ratio of the total number of failed components p_f and the total number p=100000 observed components was formed. The validity of equations (3) and (5) has been confirmed with each Monte Carlo simulation.

Inequality (2) can also be rewritten as

$$(1-\bar{x}^2)^n \geq (1-x_1^2)(1-x_2^2)...(1-x_n^2)\,, \quad (6)$$

or considering (5), it can also be rewritten as:

$$(1-(p_f\,/\,p)^2)^n \geq (1-x_1^2)(1-x_2^2)...(1-x_n^2) \quad (7)$$

The right-hand side of inequality (7) gives the actual reliability on demand of the series-parallel system in Fig.3a including components of n varieties. Because the probabilities of failure x_i characterising the separate varieties and the number of components from the separate varieties are never known, the system reliability on demand must necessarily be estimated through the left-hand side of inequality (7).

The reverse engineering of inequality (2) states that the predicted reliability on demand of a series-parallel systems based on an average probability of failure on demand $\bar{x} = p_f\,/\,p$, is higher than the actual reliability of the system. This always holds true provided that the estimate $\bar{x} = p_f\,/\,p$ is sufficiently accurate.

The significant divergence between the projected and actual system reliability on demand, caused by variability, can be remarkably pronounced, as evidenced by the following numerical examples.

Let's consider 900 valves of the same type X but of three different varieties (for example, valves from machine centres 1, 2 and 3). The valves work independently from one another. From past failure statistics, 261 of the monitored 900 valves fail to close on demand. Because only the total number of valves 900 and the total number of unreliable valves are known, the probability of failure on demand for the valves of type X will be estimated from:

$$\bar{x} = p_f\,/\,p = 261/900 = 0.29$$

Now, suppose that the series-parallel system in Fig.3b includes two valves from each of the three varieties.

The estimated system reliability based on average probability of failure on demand $\bar{x}$ becomes:

$$R_{est} = (1-(p_f\,/\,p)^2)^3 = (1-0.29^2)^3 = 0.77\,.$$

For the sake of simplicity, assume that each of the three manufacturing centers has produced 300 valves of type X, resulting in valves of three distinct varieties. Let the number of unreliable valves from these varieties be 12, 42, and 207, respectively. Consequently, the probability of failure on demand for each variety is as follows: $x_1 = 12/300 = 0.04$, $x_2 = 42/300 = 0.14$ and $x_3 = 207/300 = 0.69$, correspondingly.

As can be verified, the following expression holds true for the average probability of failure $\bar{x}$:

$$\bar{x} = (x_1 + x_2 + x_3)/3 = (0.04+0.14+0.69)/3$$
$$= 0.29 = p_f\,/\,p = 261/900$$

Suppose that valves from each variety have been used to construct the three sections arranged in series in Fig.3b. The actual (real) reliability of the series-parallel system is:

$$R_{real} = (1-0.04^2)\times(1-0.14^2)\times(1-0.69^2)$$
$$= 0.51$$

The estimated reliability on demand $R_{est} = 0.77$ is 1.51 times greater than the real reliability $R_{real} = 0.51$!

In the next example, the actual reliability of the system in Fig.4a is given by the left part of inequality (1) for m redundant components in each section in series, while the right part provides an estimate of the system reliability based on the average probability of failure on demand characterising the n different component varieties.

For $n=3$ sections with $m=3$ redundant components in each section (Fig.4b), inequality (1) becomes

$$(1-x_1^3)(1-x_2^3)(1-x_3^3) \leq$$
$$\left(1-[(x_1+x_2+x_3)/3]^3\right)^3 = \left(1-(p_f/p)^3\right)^3 \quad (8)$$

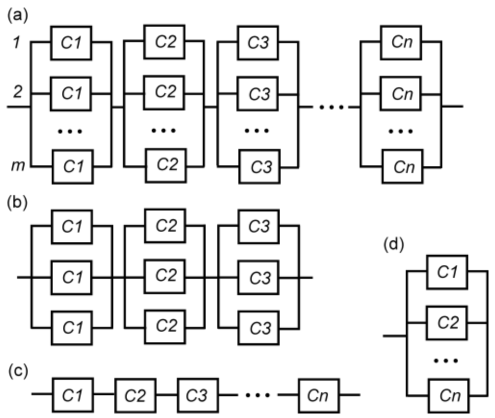

Figure 4. a) Reliability network of a series-parallel system with components from *n* varieties and *m* redundancies in each block; b) Reliability network of a series-parallel system with components from *3* varieties and *3* redundancies in each block; c) Reliability network of a system with components in series, from the same type and *n* varieties; d) Reliability network of a system with components in parallel, from the same type and *n* varieties

where the left part of (8) is the actual reliability of the system in Fig.4b while the right part is the reliability of the system in Fig.4b, estimated by using the average probability of failure on demand of the three varieties. For components of the same three varieties as in the previous example, the left-hand side of inequality (8) gives:

$$R_{real} = (1-0.04^3)\times(1-0.14^3)\times(1-0.69^3) = 0.67$$

for the real reliability of the arrangement in Fig.4b.

If the reliability of the section in Fig.4b is calculated on the basis of the average probability of failure on demand

$$\bar{x} = p_f / p = 261/900 = 0.29$$

characterising the three varieties, for the estimated system reliability on demand, the right-hand side of (8) gives:

$$R_{est} = (1-(p_f/p)^3)^3 = (1-0.29^3)^3 = 0.93\,.$$

2.3. System with components in series

If in inequality (1), we set $m=1$, the inequality transforms into:

$$(1-\bar{x})^n \geq (1-x_1)(1-x_2)...(1-x_n) \quad (9)$$

where $\bar{x} = p_f / p$ is the average probability of failure on demand. For simplicity, for systems in series, we assume that the number of components is equal to the number *n* of varieties. Clearly, the right part of inequality (9) is the actual reliability on demand of a system with *n* components logically arranged in series, of the same type *X* and *n* different varieties (Fig.4c). The left part of inequality (9) is the reliability on demand of the same system, evaluated by taking an average probability of failure on demand $\bar{x} = p_f / p$ for the components. The quantity $r_i = 1 - x_i$ in inequality (9) is the reliability on demand of the components from variety *i* (i=1,...,*n*) while $\bar{r} = 1-\bar{x}$ is the average reliability on demand characterising all components of the given type *X*. Noticing that

$$1-\bar{x} = 1-\frac{x_1+x_2+...+x_n}{n} = \frac{(1-x_1)+(1-x_2)+...+(1-x_n)}{n}$$

inequality (9) can also be rewritten as

$$\left(1-\frac{p_f}{p}\right)^n = \left(\frac{r_1+r_2+...+r_n}{n}\right)^n \geq r_1 r_2 ... r_n \quad (10)$$

which is the classical Arithmetic mean – Geometric mean (AM-GM) inequality (Steele,2004). Inequality (10) clearly highlights the overestimation of system reliability on demand when using the average reliability on demand for components from different varieties.

This conclusion could have been reached directly, without addressing the special case of inequality (9), had reverse engineering been applied to the AM-GM inequality (10).

Consider three valves of the same three varieties as in the previous example, with the valves logically arranged in series. As in the previous examples, the varieties have reliabilities on demand of $0.96 = 1 - 0.04$, $0.86 = 1 - 0.14$ and $0.31 = 1 - 0.69$, respectively, and operate independently of each other.

The right-part of inequality (10) gives the actual reliability of the series arrangement which is a product of the reliabilities on demand of the independently operating valves.

Therefore, the actual reliability of the section with these three valves is:

$$R_{real} = 0.96 \times 0.86 \times 0.31 = 0.25$$

The left part of inequality (10) gives the system reliability calculated on the basis of the average reliability on demand $\bar{r}$ characterising the three varieties:

$$\bar{r} = 1 - p_f / p = 1 - 261/900 = 0.71$$

Note that $\bar{r} = (0.96 + 0.86 + 0.31)/3 = 0.71$

The estimated system reliability based on average reliability on demand from the left part of inequality (10) is:

$$R_{est} = \bar{r}^3 = 0.71^3 = 0.36 .$$

The estimated value $R_{est} = 0.36$ is 1.44 times greater than the real reliability $R_{real} = 0.25$ of the section!

2.3. System with components logically arranged in parallel

For simplicity, for systems in parallel, we also assume that the number of components is equal to the number n of varieties.

Suppose that r_i in the AM-GM inequality (10) are set to be the probabilities of failure x_i of the different component varieties. Inequality (10), then transforms into

$$(p_f / p)^n = \left(\frac{x_1 + x_2 + ... + x_n}{n} \right)^n \geq x_1 x_2 ... x_n \tag{11}$$

The right-hand side of inequality (11) then can be interpreted as the probability of failure of a system where all independently working components are logically arranged in parallel (Figure 4d). A system consisting of n independently working components arranged in parallel is in a failed state when all components are in failed state. The left-hand side of inequality (11) is the probability of failure of the system in parallel, estimated on the basis of the average probability of failure of the components from different varieties. The physical interpretation of inequality (11) suggests that for parallel systems, when estimates are derived from the average probability of component failure on demand, the projected probability of system failure exceeds the actual value.

Even for statistically independent components, this is not a reliable method for assessing system reliability.

Based on the findings from the reverse engineering of inequality (1), we can draw the following conclusions. The prevailing methodology for predicting system reliability on demand, which relies on average component reliabilities on demand for components of different varieties but the same type, is fundamentally flawed due to component variability.

If there were no variability in the reliabilities of components of the same type, inequalities (10-11) would become equalities, and there would be no discrepancy between the estimated and the actual system reliability.

The larger the deviations of the component reliabilities from the average value, the stronger the inequalities (10-11).

Deviations in reliabilities on demand from the average value are inevitable, primarily due to differences in age, working conditions, material, and manufacturing flaws. Consequently, discrepancies between the predicted reliability on demand and the actual value will always exist.

Assuming average reliability on demand for components of a particular type however, can still provide valuable insights, if the scope is confined to a comparative analysis that ranks competing system designs.

2.4 Evaluating system reliability related to a specified time interval

It is important to discuss also the impact of variability on reliability predictions when evaluating reliability over a specified time interval.

Consider a system with components of the same type and n varieties, logically arranged in parallel. Suppose that each component variety is characterised by a constant hazard rate λ_i, $i = 1,...,n$. Consequently, the time to failure distribution for a component is the negative exponential time to failure distribution:

$$F(t) = 1 - \exp(-\lambda_i t) \quad (12)$$

where t is the time interval and $F(t)$ is the probability that the component will fail before time t.

Consider n components logically arranged in parallel (Figure 4d), each of which is from a different variety. According to the system reliability theory, the actual probability of failure F_{s1} of the system before time t is given by

$$F_{s1} = (1 - e^{-\lambda_1 t}) \times (1 - e^{-\lambda_2 t}) \times ... \times (1 - e^{-\lambda_n t}) \quad (13).$$

(The system is in a failed state only if all components are in a failed state at time t).

If the probability of failure or the system reliability is calculated based on the average hazard rate:

$$\bar{\lambda} = \frac{\lambda_1 + \lambda_2 + ... + \lambda_n}{n} \quad (14)$$

the probability of system failure before time t is given by

$$F_{s2} = (1 - e^{-\bar{\lambda} t})^n \quad (15)$$

It can be shown that the inequality:

$$(1 - e^{-\bar{\lambda} t})^n \geq (1 - e^{-\lambda_1 t}) \times (1 - e^{-\lambda_2 t}) \times ... \times (1 - e^{-\lambda_n t}) \quad (16)$$

always holds. Inequality (16) can be proved by taking logarithms from both sides, showing that the right-hand side is a concave function and applying the Jensen's inequality. The proof of inequality (16) is similar to the proof of inequality (1) given in the Appendix and because of space constraints, the details will be omitted.

As a result, using average hazard rate to estimate the reliability of systems in parallel always leads to overestimating the probability of failure of the system.

Despite their popularity, system reliability predictions based on average failure rates (e.g., MIL-STD-1629A, 1977) have very serious shortcomings. The failure of this approach to generate accurate system reliability predictions led to growing disillusionment among researchers and practitioners. As a result, many abandoned the use of failure rate-based reliability predictions.

For systems with components logically arranged in series however, the system reliability estimated on the basis of an averaged hazard rate is exactly equal to the actual reliability of the system.

Indeed, let $\lambda_1, \lambda_2, ..., \lambda_n$ be the constant hazard rates characterising the n component varieties. Since the reliability of a single component of variety i is given by $R_i = \exp(-\lambda_i t)$, the reliability of the system based on n components, each of which is of different variety, is given by the expression

$$R_{s1} = \exp(-\lambda_1 t) \times \exp(-\lambda_2 t) \times ... \times \exp(-\lambda_n t) = \exp(-[\lambda_1 + \lambda_2 + ... + \lambda_n] t) \quad (17)$$

If the reliability of the system in series is calculated on the basis of the average hazard rate of the n varieties (given by equation (14)), the system reliability is given by

$$R_{s2} = \exp(-\bar{\lambda} t) \times \exp(-\bar{\lambda} t) \times ... \times \exp(-\bar{\lambda} t) = \exp(-[\lambda_1 + \lambda_2 + ... + \lambda_n] t) \quad (18)$$

Since $R_{s1} = R_{s2}$, working with an average hazard rate for components of a particular type neither overestimates nor underestimates the predicted system reliability.

3. Reverse engineering of sub-additive inequalities

3.1 Link between sub-additive algebraic inequalities and concave functions

A function *f(x)* is said to be sub-additive if it satisfies the following inequality for any set of non-negative values $x_1, x_2, ..., x_n$:

$$f(x_1 + x_2 + ... + x_n) \le f(x_1) + f(x_2) + ... + f(x_n) \quad (19)$$

A key property associated with sub-additive inequalities can now be presented (Alsina and Nelsen, 2010). If a function *f(x)*, with a domain $[0,\infty)$ and range $[0,\infty)$, is concave, then the function exhibits sub-additive behaviour and satisfies inequality (19).

Let an additive controlling factor x be divided into a number of non-negative parts (segments) $x_1, x_2, ..., x_n$;

$$x = x_1 + x_2 + ... + x_n;\ n \ge 2 \quad (20)$$

Additive quantities vary with the size of a system, increasing or decreasing as the system's size changes (e.g., mass, volume, energy, heat, power, distance, area, etc.). On the other hand, non-additive quantities remain constant regardless of changes to the size of the system (e.g. pressure, concentration, temperature, etc.).

The outputs corresponding to the additive factor x and the individual segments x_i are also assumed to be additive quantities, denoted by $f(x)$ and $f(x_i)$ correspondingly. The sub-additive inequality (19) has a large range of potential applications. Let the function *f(x)* quantify the output associated with a specific additive factor of magnitude *x*, and x_i be the sizes of the segments into which this factor has been divided. For a concave function *f(x)* with domain $[0,\infty)$ and range $[0,\infty)$, splitting the factor with magnitude *x* leads to a higher total output.

3.2 Reverse engineering of sub-additive inequalities based on concave power laws

Reverse engineering of sub-additive inequalities used for process optimisation will be illustrated on concave power-law dependences of the type:

$$y = ax^p \quad (21)$$

where $p > 0$ is an exponent and $a > 0$ is a constant, *x* is the magnitude of an additive controlling factor and the output 'y' is also an additive quantity. If the power p in dependence (21) is a number from the interval (0,1), the power-law dependence (21) is concave.

A concave power law is one in which the rate of change of the output decreases as the controlling factor increases. A graph of a concave power law function is shown in Fig.5.

In the subsequent sections, we demonstrate that the reverse engineering of sub-additive inequalities, which are based on concave power laws of the type presented with dependence (21), can be used to optimise process performance.

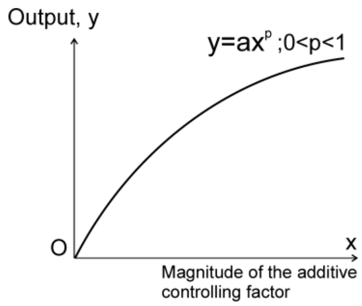

Figure 5. A concave power law.

The quantity *y* is a concave function of *x* ($x \ge 0$) because the second derivative of *y* is negative for $0 < p < 1$. Consequently, for $0 < p < 1$, the following sub-additive inequality holds:

$$a(x_1 + x_2 + ... + x_n)^p \le ax_1^p + ax_2^p + ... + ax_n^p \quad (22)$$

Note that the additivity of the controlling factor *x* and the outputs ax_i^p are necessary conditions for the reverse engineering of inequality (22). Otherwise, the expression $x = x_1 + x_2 + ... + x_n$ and the sum $ax_1^p + ax_2^p + ... + ax_n^p$ would not possess a valid physical interpretation.

The advantage of the sub-additive inequality (22), based on a concave power law, is that its reverse engineering is simple and can be made in diverse application areas.

Inequality (22) can also be generalised for two sets including different number of equal-sized segments. Suppose that $x = x_{1u} + x_{2u} + ... + x_{mu}$ where $x_{iu} = x/m$ and $x = x_{1v} + x_{2v} + ... + x_{nv}$ where $x_{iv} = x/n$ and also $m < n$. It can then be shown that

$$ax_{1u}^p + ax_{2u}^p + ... + ax_{mu}^p \leq ax_{1v}^p + ax_{2v}^p + ... + ax_{nv}^p \tag{23}$$

The proof of inequality (23) has been provided in the Appendix.

3.3 Generalisation of inequality (23) for unequal-sized segments

Inequality (23) can also be generalised for two sets including the same number of unequal-sized segments. Suppose that

$$x_{1u} \geq x_{2u} \geq ... \geq x_{nu} \tag{24}$$

$$x_{1v} \geq x_{2v} \geq ... \geq x_{nv} \tag{25}$$

and also

$$x_{1u} + x_{2u} + ... + x_{nu} = x_{1v} + x_{2v} + ... + x_{nv}.$$

Definition: It is said that segments x_{iu} majorise the segments x_{iv} if the following conditions are met (Marshall et al., 2011):

$$x_{1u} \geq x_{1v};\ x_{1u} + x_{2u} \geq x_{1v} + x_{2v}; ...;$$
$$x_{1u} + x_{2u} + ... + x_{n-1u} \geq x_{1v} + x_{2v} + ... + x_{n-1v};$$
$$x_{1u} + x_{2u} + ... + x_{nu} = x_{1v} + x_{2v} + ... + x_{nv}; \tag{26}$$

If the segments x_{iu} majorise the segments x_{iv} where $i = 1,...,n$, then the inequality

$$ax_{1u}^p + ax_{2u}^p + ... + ax_{nu}^p \leq ax_{1v}^p + ax_{2v}^p + ... + ax_{nv}^p \tag{27}$$

holds. A proof of inequality (27) has also been provided in the Appendix.

Inequality (27) creates the possibility of increasing the effect from an additive resource factor by reallocation of the resource segments only, without resorting to any extra segmentation.

4. Reverse engineering of sub-additive inequalities for optimising engineering processes

4.1 Enhancing the yield from processes described by a concave power law

The application of reverse engineering of sub-additive inequalities based on concave power laws will be illustrated by using an additive controlling factor. Here is a list of applications involving concave power laws:

(i) The rate of natural resource extraction, such as oil or minerals, often follows a concave power law, as the remaining reserves become increasingly difficult and expensive to access.

(ii) Economic growth in mature economies often exhibits concave power law, as diminishing returns on investment and resource constraints limit the rate of expansion.

(iii) The growth of organisms, including plants and animals, often follows a concave power law with respect to the mass of the organism. The rate of growth decreases as the organism reaches maturity.

(iv) The growth of a material's strength or other properties as a function of the amount of reinforcing substance added can often be described well by a concave power law, due to saturation.

(v) Due to limited solubility, the growth of gas solubility in a liquid as a function of the partial pressure of the gas, can be approximated well by a concave power law.

(vii) The growth of skill or knowledge acquisition as a function of time or practice, often exhibits concave power law as individuals approach the limits of their potential.

Suppose that the controlling factor x is interpreted as an investment in a particular enterprise. The yield from the investment increases according to the concave power law (21), where x is the magnitude of the investment (an additive quantity) and y is the yield from the investment (also an additive quantity).

A reverse engineering of the sub-additive inequality (22) based on the concave power law (21) shows that the yield from the investment can be increased if the total investment $x = x_1 + x_2 + ... + x_n$ is split into n segments $x_1, x_2, ..., x_n$ and the separate segments x_i are invested in n parallel enterprises of the same type, bringing yields ax_i^p.

Here is a simple numerical illustration. Suppose that the profit y from an investment of size x is given by the concave power law:

$$y = 210.2 \times x^{0.3} \quad (28)$$

Investing $2 million in a single enterprise generates the profit:

$$y = 210.2 \times 2000000^{0.3} = 16328$$

Now suppose that two parallel investments are allocated in two enterprises of the same type as follows: $1.8 million in the first enterprise and $0.2 million in the second enterprise. According to the sub-additive inequality (22), this allocation yields a larger profit. Indeed, the profit generated from the two parallel investments is:

$$y = 210.2 \times 1800000^{0.3} + 210.2 \times 200000^{0.3}$$
$$= 24004$$

Now suppose that the investment in each enterprise has been reallocated in such a way that according to inequalities (26), it is majorised by the previous investment: for example, $1 million in the first enterprise and $1 million in the second enterprise:

$$\$1.8 > \$1;\ \$1.8 + \$0.2 = \$1 + \$1$$

According to inequality (27), this reallocation of the investment should yield even larger profits. Indeed, the combined profit from the reallocated investment is:

$$y = 210.2 \times 1000000^{0.3} + 210.2 \times 1000000^{0.3}$$
$$= 26525$$

The obtained combined profit from the segmented investment with distribution (($1 million, $1 million) is approximately 1.62 times greater than the profit from the single enterprise investment and 1.1 times greater than the segmented investment with distribution ($1.8 million, $0.2 million). This significant increase of profit demonstrates the considerable impact of segmenting the additive controlling factor following the approach based on inequality (27).

4.2 Improving the absorption effectiveness

Absorption capacity is the ability of a material to assimilate another substance within its structure. It is often quantified as the maximum amount of substance that can be absorbed per unit mass or volume of the absorbent material.

In some cases, the relationship between absorption capacity and volume of absorbent is non-linear. This can be due to factors such as porosity, surface area, affinity for the substance being absorbed, as well as specific interactions between the absorbent and the absorbed substance. A non-linear dependence expressing the absorption capacity as a function of the volume of absorbent can be described by the power law function:

$$C(x) = ax^b$$

Here, C(x) represents the quantity of absorbed substance (absorption capacity) corresponding to volume x of absorbent, and a and b are constants that depend on the specific absorbent material and the substance being absorbed.

If the absorption capacity saturates or levels off at higher volumes, the exponent b would be less than 1. The volume x of absorbent is an additive quantity as well as the quantity C(x) of absorbed substance. Segmenting the volume x of the absorbent into smaller volumes $x_1, x_2, ..., x_n$ ($x = \sum_{i=1}^{n} x_i$) through which the substance is filtered, results in quantities:

$$C_1 = ax_1^b,\ C_2 = ax_2^b, ..., C_n = ax_n^b$$

of absorbed substance.

Consider the following specific numerical values for the coefficient a and the power b: $a = 0.12$ (grams of absorbed substance per one gram of absorbent) and $p = 0.4$.

For a single filter containing 90000g of absorbent, the absorbed substance is $C = 0.12 \times 90000^{0.4} = 11.5g$.

Suppose that the 90000g absorbent is segmented into three parallel filters containing 50000g, 20000g and 20000g of absorbent, respectively.

The total amount C of absorbed harmful substance is obtained by adding the quantities absorbed by each filter:

$$C = 0.12 \times 50000^{0.4} + 0.12 \times 20000^{0.4} + 0.12 \times 20000^{0.4} = 21.7g$$

The total amount of absorbed harmful substance is 1.89 times larger than the initial amount of absorbed substance!

4.3 Increasing the flow rates of systems of pumps

The flow rate (y) of a pump often follows a concave power law relationship with respect to the pump's power consumption (P).

$$y = aP^k \qquad (29)$$

In equation (29), a and k are constants that depend on the specific pump while P is the power consumed by the pump. The constant k is a fractional exponent between 0 and 1. Relationship (29) indicates that the speed of increasing the flow decreases as the pump operates at higher power levels.

The sub-additive inequality (23) based on the concave power law (29) can be used to increase the flow rate of a system of m identical pumps working in parallel, with power consumptions:

$$P_1,\ P_2,...,P_m \ (P_i = P/m)$$

According to the sub-additive inequality (23), the flow rate of the system of pumps at the same consumed power $P = \sum_{i=1}^{m} P_i$ can be increased by increasing the number of pumps while simultaneously reducing the power of a single pump such that we have n pumps $(n > m)$ with smaller power consumptions:

$$Q_1,\ Q_2,...,Q_n \ (Q_i = P/n).$$

The total power consumption of the pumps from the new design option is the same: $P = \sum_{i=1}^{m} P_i = \sum_{i=1}^{n} Q_i$.

According to the properties of sub-additive functions

$$aP_1^k + aP_2^k + ... + aP_m^k \le aQ_1^k + aQ_2^k + ... + aQ_n^k \qquad (30)$$

which creates the possibility of increasing the flow rate of the system of m pumps by including a larger number $n>m$ of pumps with smaller power, whose combined consumed power is equal to the power P of the initial m pumps.

According to the sub-additive inequality (27) however, the flow rate of the system of m pumps at the same consumed power $P = \sum_{i=1}^{m} P_i$ can be increased without increasing the number of pumps. This can be done if the powers $Q_1,\ Q_2,...,Q_m$ of the new pumps are selected such as they are majorised by the powers $P_1,\ P_2,...,P_m$ of the original pumps.

Suppose that $P_1 \ge P_2 \ge ... \ge P_m$. The powers of the new set of pumps are then selected such that $Q_1 \ge Q_2 \ge ... \ge Q_m$ and

$$P_1 \ge Q_1;\ P_1 + P_2 \ge Q_1 + Q_2;...;$$
$$P_1 + P_2 + ... + P_{m-1} \ge Q_1 + Q_2 + ... + Q_{m-1};$$
$$P_1 + P_2 + ... + P_m = Q_1 + Q_2 + ... + Q_m; \qquad (31)$$

If conditions (31) are present, according to inequality (27), the following inequality holds:

$$aP_1^k + aP_2^k + ... + aP_m^k \le aQ_1^k + aQ_2^k + ... + aQ_m^k \qquad (32)$$

which creates the possibility of increasing the flow rate of a system of n pumps by selecting the same number of pumps with the same total consumed power.

5. Conclusions

1. A key inequality related to reliability of common systems has been stated and proved.
2. The reverse engineering of the stated inequality revealed a fundamental flaw in the current approach related to predicting system reliability. Because of the variability of components from a particular type, using average

component reliabilities on demand to evaluate reliability of series-parallel systems, systems in series and systems in parallel is a fundamentally flawed approach, even for systems with components working independently from one another.

3. Because of the variability of components from a particular type, using average hazard rates to evaluate the reliability of systems with components logically arranged in parallel, is also a fundamentally flawed approach, even for systems with components working independently from one another.
4. The reverse engineering of sub-additive inequalities for processes described by a concave power law has led to strategies for enhancing the performance of various industrial processes.
5. Several properties of sub-additive inequalities based on a concave power law have been discussed and proven.

APPENDIX

Proof of inequality (1)

From the basic properties of the concave functions $f(x)$ and $g(x)$: $f[\lambda x+(1-\lambda)y]\geq\lambda f(x)+(1-\lambda)f(y)$, and $g[\lambda x+(1-\lambda)y]\geq\lambda g(x)+(1-\lambda)g(y)$ where $0\leq\lambda\leq1$, it can be shown easily that the sum $h(x)=f(x)+g(x)$ of two concave functions $f(x)$ and $g(x)$ is a concave function. By induction, it can be deduced that the sum of *n* concave functions is also a concave function.

Inequality (1) can be proved by observing that the sum of the logarithms: $z=\ln(1-x_1^m)+\ln(1-x_2^m)+...+\ln(1-x_n^m)$ is a concave function because it is a sum of the *n* concave functions $z_1=\ln(1-x_1^m);\ z_2=\ln(1-x_2^m);...;z_n=\ln(1-x_n^m)$. The functions $z_i=\ln(1-x_i^m)$ are concave because their second derivatives are all negative:

$$\frac{\partial^2 z_i}{\partial x_i^2}=-\frac{m(m-1)x_i^{m-1}(1-x_i^m)+m^2x_i^{2(m-1)}}{(1-x_i^m)^2}<0.$$

considering that $m-1\geq0$ and $1-x_i^m>0$.

Let w_i be weights defined such that $w_1=w_2=...=w_n=1/n$. According to the Jensen's inequality (Steele, 2004), if $\bar{x}=w_1x_1+w_2x_2+...+w_nx_n$, the following inequality holds for a concave function:

$$w_1\times\ln(1-x_1^m)+w_2\times\ln(1-x_2^m)+...+w_n\times\ln(1-x_n^m)\leq\ln(1-(w_1x_1+w_2x_2+...+w_nx_n)^m)\quad\text{(A1)}$$

Inequality (A1) can be rewritten as:

$$\ln[(1-x_1^m)(1-x_2^m)...(1-x_n^m)]\leq\ln[1-((x_1+x_2+...+x_n)/n)^m]^n\quad\text{(A2)}$$

Since the exponential function e^x is strictly increasing, according to the properties of inequalities, the direction of inequality (A2) will not change if both sides of (A2) are exponentiated:

$$\exp(\ln[(1-x_1^m)(1-x_2^m)...(1-x_n^m)])\leq\exp(\ln[1-((x_1+x_2+...+x_n)/n)^m]^n)\quad\text{(A3)}$$

which yields inequality (1).

Proof of inequality (23).

Proving inequality (23) is equivalent to proving the inequality

$$m\times a(x/m)^p\leq na(x/n)^p\quad\text{(A4)}$$

which is equivalent to proving the inequality

$$1/m^{p-1}\leq1/n^{p-1}\quad\text{(A5)}$$

which is equivalent to the inequality

$$m^{1-p} \le n^{1-p} \tag{A6}$$

Since $1-p>0$ and $m<n$, inequality (A6) is always true and this completes the proof of inequality (23).

Proof of inequality (27).

First, if for any i, $x_{iu} = x_{iv}$, then the inequality (27) will not be affected if x_{iu} and x_{iv} are removed. As a result, without loss of generality, it can be assumed that $x_{iu} \ne x_{iv}$, for all i.

Consider the slope of the secant $\dfrac{ax_u^{\ p} - ax_v^{\ p}}{x_u - x_v}$ through the points (x_u, ax_u^p) and (x_v, ax_v^p). The power law function $y = ax^p$ is concave and monotonically increasing function in x and considering also conditions (24) and (25), this implies that the following property holds: (see Fig.A1):

$$k_i = \tan(\alpha) = \frac{ax_{iu}^p - ax_{iv}^p}{x_{iu} - x_{iv}} \le \frac{ax_{i+1u}^p - ax_{i+1v}^p}{x_{i+1u} - x_{i+1v}} = k_{i+1} = \tan(\beta) \tag{A7}$$

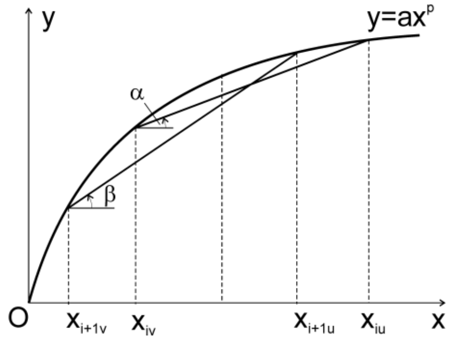

Figure A1. The key property (A7) for the concave power law function $y = ax^p$.

Let $X_{0u} = X_{0v} = 0$ and $X_{iu} = x_{1u} + x_{2u} + ... + x_{iu}$, $X_{iv} = x_{1v} + x_{2v} + ... + x_{iv}$, $i = 1,...,n$

From the majorisation property (26), it follows that $X_{iu} \ge X_{iv}$ for $i = 1,...,n-1$ and $X_{nu} = X_{nv}$

Proving inequality (27) is equivalent to proving the inequality $\sum_{i=1}^{n} [ax_{iu}^p - ax_{iv}^p] \le 0$.

From (A7), it follows that

$$\sum_{i=1}^{n} [ax_{iu}^p - ax_{iv}^p] = \sum_{i=1}^{n} k_i (x_{iu} - x_{iv}) \tag{A8}$$

Since $x_{iu} = X_{iu} - X_{i-1u}$ and $x_{iv} = X_{iv} - X_{i-1v}$, the sum in the right-hand side of (A8) can be presented as

$$\sum_{i=1}^{n} k_i (x_{iu} - x_{iv}) = \sum_{i=1}^{n} k_i (X_{iu} - X_{i-1u} - (X_{iv} - X_{i-1v})) = \sum_{i=1}^{n} k_i (X_{iu} - X_{i-1u}) - \sum_{i=1}^{n} k_i (X_{iv} - X_{i-1v}) \tag{A9}$$

In turn, the sum in the right hand side of equation (A9) can be presented as:

$$\sum_{i=1}^{n} k_i (X_{iu} - X_{i-1u}) - \sum_{i=1}^{n} k_i (X_{iv} - X_{i-1v}) = k_n (X_{nu} - X_{nv}) - k_1 (X_{0u} - X_{0v}) + \sum_{i=1}^{n-1} (k_i - k_{i+1})(X_{iu} - X_{iv})$$

Considering that $k_n (X_{nu} - X_{nv}) = 0,\ k_1 (X_{0u} - X_{0v}) = 0$, the sum becomes:

$$\sum_{i=1}^{n} k_i (X_{iu} - X_{i-1u}) - \sum_{i=1}^{n} k_i (X_{iv} - X_{i-1v}) = 0 - 0 + \sum_{i=1}^{n-1} (k_i - k_{i+1})(X_{iu} - X_{iv})$$

As a result, the relationship: $\sum_{i=1}^{n} k_i (x_{iu} - x_{iv}) = \sum_{i=1}^{n-1} (k_i - k_{i+1})(X_{iu} - X_{iv})$

has been established. Since $X_{iu} - X_{iv} \geq 0$ and $k_i - k_{i+1} \leq 0$, it follows that $\sum_{i=1}^{n} k_i (x_{iu} - x_{iv}) \leq 0$, which proves inequality (27).

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