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Numerical cognition and mathematical realism

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1. Introduction

Recently there has been a lot of discussion on evolutionary debunking arguments, especially in the moral domain (*e.g.*, Street 2006; see Vavova 2014 for review), and lately also in the mathematical domain. On the face of it, evolved features of numerical cognition support realism about numbers. For instance, Joyce (2006, 135) has claimed that truth and fitness in the mathematical case, unlike in the moral case, do not come apart: "we have no grasp of how this belief [the belief that 1 + 1 = 2] might have enhanced reproductive fitness independent of assuming its truth". By contrast, Clarke-Doane (2012) has argued that the evolutionary challenge for moral realism applies equally to mathematical realism: if evolutionary debunking arguments can successfully undermine moral realism, they can also undermine mathematical realism.

Surprisingly, in these discussions about mathematical realism and evolution, there has been no attention paid to evolved features of numerical cognition in humans and other animals. For instance, Sinnott-Armstrong (2006, 43) writes, "People evolved to believe that 2 + 3 = 5, because they would not have survived if they had believed that 2 + 3 = 4, but the reason why they would not have survived then is that it is true that 2 + 3 = 5. The same goes for the belief that wild animal bites hurt. In such cases, the truth of the belief explains why it is useful to believe it." As I will demonstrate further on, the belief that 2 + 3 = 5 lies outside of the scope of evolved numerical cognition — without the help of culturally developed tools such as counting, we would not know whether 2 + 3 = 4, 5, or even 8, although we'd know that 2 + 3 is definitely not 100.

To find out whether evolved numerical cognition supports antirealism or realism about numbers, philosophers need to move beyond broad generalizations, and look at what the empirical data support. Fortunately, we are in an excellent epistemic situation to do this. Numerical cognition ranks among the most extensively studied higher cognitive functions in animals, so we have at our disposal a wealth of empirical data that is potentially relevant for philosophical arguments about the compatibility of numerical cognition with realism or antirealism.

In this paper, I will look in detail at the functional properties of evolved numerical cognition and examine whether this supports realism or antirealism. In section 2, I briefly review evolutionary arguments and realism about numbers. Section 3 looks at the nuts and bolts of evolved numerical cognition, showing that there is overwhelming empirical support for the claim that a wide range of animals (including humans) have an evolved propensity to represent discrete magnitudes in their environment ("numerosities"). Section 4 presents a positive program, where I explore how realism about numbers could be true, given what we know about evolved numerical cognition. I formulate an argument for mathematical realism as an inference to the best explanation for functional features of numerical cognition. Section 5 concludes that evolved features of mathematical cognition can be explained in a realist way, challenging Clarke-Doane's evolutionary argument against mathematical realism.

2. Evolution and realism about numbers

An enduring debate in the philosophy of mathematics concerns the ontological status of numbers, such as 2, π, and 34,295.17. Realists (*e.g.*, Baker 2005) argue that numbers exist mind-independently, whereas antirealists (or nominalists) propose that numbers do not exist apart from our own minds (*e.g.*, Leng 2005). Among the varieties of mathematical realism, the most influential remains platonism, which specifies that abstract entities are nonphysical; they are not located in space-time, and cannot stand in causal relation to physical states of affairs.

What reasons do we have to assume that such acausal entities exist? Realism about numbers, as well as about moral facts and other putative abstract entities, has recently come under pressure from evolutionary debunking arguments. Such arguments have the following general form:[1]

1. We have an evolved propensity to believe that p, where p is a belief about abstract objects in domain D, because this belief conferred an adaptive advantage to our ancestors.
2. Even if it were the case that ¬p in some realist sense, it would still have been more adaptively advantageous to believe that p.
3. (from 1 and 2) We would have believed that p regardless of the actual truth value of p.
4. Therefore, our belief that p does not track a mind-independent property of abstract objects in D.

Premise 1 accords an important causal role to evolution in the formation of our beliefs: evolution has played a "tremendous role" (Street 2006, 109) in shaping our belief-forming attitudes. Remarkably, premise 1 is seldom argued for in detail, yet, as Kahane (2011, 111) puts it, this premise makes an "extremely ambitious empirical claim", especially given that the data to secure this premise are lacking for many domains in which evolutionary debunking arguments are proposed. The next section will consider the evolutionary origins of numerical cognition to evaluate the plausibility of premise 1.

Premise 2 holds that a realm of abstract entities (in the moral, mathematical, religious, etc. domain), if it exists, does not influence the evolutionary trajectory of the cognitive faculties that represent them. Therefore, (3) evolved beliefs are insensitive to the truth-value of abstract objects, which leads to the conclusion (4) that our evolved beliefs do not track facts about abstract objects. Appealing to parsimony, antirealists argue that antirealism is more compatible with the evolutionary history of our beliefs than realism.

1. This schema captures the structure of evolutionary debunking arguments against realism. For a more general schema for evolutionary debunking arguments, see Kahane (2011).

Clarke-Doane (2012) has developed an evolutionary debunking argument specifically aimed at mathematical realism. It relies on intuitions elicited by the following toy example: Suppose that a lion is hiding behind bush A, and a second lion is hiding behind bush B. Human ancestor P believes that 1 + 1 = 2 and flees. Human ancestor Q believes that 1 + 1 = 0 and stays. Prima facie, we can argue that because 1 + 1 = 2, ancestor P has an advantage over Q. This arithmetical truth figures in the evolutionary explanation, therefore, our belief that 1 + 1 = 2 tracks a mathematical truth. However, Clarke-Doane (2012) proceeds with a counterfactual scenario, imagining a world where 1 + 1 really equals 0. Realistically construed, 1 + 1 = 0 speaks about numbers. Now suppose we hold the first-order logical truths constant, but change the mathematical truth to 1 + 1 = 0. In that case, ancestor Q, although she now has the correct mathematical belief, would still get eaten, as there would still be one lion and another lion waiting for her. Accordingly, numerical truths do not play a relevant role in this evolutionary scenario — if the numerical truths had been different, but the first-order properties remained the same, it would have been more adaptively advantageous to believe that 1 + 1 = 2. In the next section, I examine the psychological literature on numerical cognition to assess whether premise 1 holds for numbers. In section 4, I scrutinize Clarke-Doane's argument in more detail. I argue that the current cognitive scientific literature on numerical thinking gives us no good reasons to believe premise 2 is true, which opens the possibility of a realist understanding of this literature.

3. Evolved numerical cognition

3.1 An evolved ability to represent numerosities

A growing body of experimental and neuropsychological literature indicates that animals have an evolved ability to detect discrete magnitudes in their environment. Cognitive scientists distinguish between *numerosities*, the concrete, discrete magnitudes that animals represent, and *numbers*, the abstract entities that are studied by mathematicians and philosophers of mathematics (De Cruz et al. 2010). Numerosities are not numbers, and so their properties do not directly bear on the question of whether numbers exist. However, without numerosities it is doubtful that we would be able to represent any numbers at all. Studies of patients with brain damage (*e.g.*, Dehaene & Cohen 1997) that affects their ability to represent numerosities, and of children with developmental dyscalculia (Mussolin et al. 2010), suggest that symbolic arithmetic, including simple arithmetical operations such as addition and subtraction, crucially depends on our ability to represent numerosities. The majority of cognitive scientists believe that cultural numerical representations are understood by being mapped onto numerosities (see Ansari 2008 for a review, but see Rips et al. 2008 for a dissenting voice). They reach this conclusion by observing that nonsymbolic and symbolic representations of numbers activate similar brain areas. For example, the spoken word 'three', the Arabic digit 3, and a collection of three items all activate the intraparietal sulci, which are implicated in a variety of numerical tasks (Eger et al. 2003). Thus numerosities are of crucial importance to understand how we represent numbers, even if the latter are culturally contingent, *e.g.*, not all languages have systems of natural numbers (see De Cruz 2008 for review).

Mammals, birds, amphibians, and even insects can distinguish between small numerosities, consisting of up to three (sometimes four) entities (*e.g.*, Dacke & Srinivasan 2008). This quick and unlearned ability to enumerate small collections, *subitizing*, has been found in all animals tested for it, including human newborns (Antell & Keating 1983). Adults are faster and less error-prone when calculating with numerosities that lie within the subitizing range than with larger numerosities (Revkin et al. 2008). When prevented from counting or using other symbolic representations of number, Western adults are unable to calculate precisely with numerosities > 3. For instance, adults prevented from subvocal counting (by having to say 'the' aloud with each key press) become increasingly imprecise when having to estimate larger numbers of key presses to make (Whalen et al. 1999).

Higher magnitudes are represented approximately, and can be distinguished only with a large enough ratio difference; the higher the numerosities, the larger the ratio difference needs to be (Xu & Spelke 2000). For instance, chicks can discriminate between collections of 2 and 3 items, but not between 3 and 4, or between 4 and 6 (Rugani et al. 2008). Animals and infants can also perform operations such as addition and subtraction with small numerosities: babies look longer when 1 + 1 = 1 than when 1 + 1 = 2 (Wynn 1992). Dogs have similar abilities (West & Young 2002). Animals can also perform addition and subtraction on larger numbers, but then they can predict results only approximately; *e.g.*, rhesus monkeys cannot exactly predict that 4 + 4 = 8, but they can pick out 8 when presented with possible solutions 2, 4, and 8 (Cantlon & Brannon 2007a).

For a long time, behavioral biologists assumed that numerical cues were a last resort on which animals rely only if no other information is available. However, when monkeys can choose between different types of cues, such as color, size, shape, and numerosity, to perform a matching task, they prefer numerical cues if the ratio differences are high enough to allow them to discriminate between different sets of items (Cantlon & Brannon 2007b). Under more naturalistic conditions, animals spontaneously rely on numerical information to guide a wide range of adaptive decisions, such as where to feed, how to aggregate in social groups, or whether or not to attack. Red-backed salamanders, when offered the choice between a tube with 2 live flies and one with 3 live flies, select the larger quantity (Uller et al. 2003). Juvenile guppies raised in total isolation can distinguish between groups composed of 1, 2, or 3 fish, and show an innate preference for larger shoals (Bisazza et al. 2010). Wood ducks, of a brood-parasitic species that lays its eggs in nests of other birds, use clutch size to guide their choice of their host's nest. When given the choice between pairs of nests with different numbers of eggs (20, 15, 10, or 5), females choose the nest with the smaller clutch, presumably because in such a nest their offspring will likely receive more care (Odell & Eadie 2010). McComb et al. (1994) conducted an experiment with free-ranging lions in Serengeti National Park, Tanzania, hiding a tape recorder that played the roars of either 1 or 3 unfamiliar individuals. As lionesses recognize each other's voices, the recordings mimicked unfamiliar intruders into the territory. Lionesses were more likely to approach the tape recorder if the members of their own pride that were present outnumbered the recorded number of individuals (*e.g.*, 6 of the own pride versus 3 recorded voices), presumably because the probability of winning a potentially fatal confrontation was higher in such cases.

In mammals, numerical cognition depends on specialized areas in the neocortex, including the bilateral intraparietal cortex, angular gyrus, and prefrontal cortex (see Nieder & Dehaene 2009 for review). The brain areas involved in recognizing numerosities and performing arithmetic are similar in rhesus monkeys, three-month-old infants, young children, and numerate adults (Izard et al. 2008). This suggests the continued importance of evolved neural structures in mature mathematical cognition. People with developmental damage to the intraparietal cortex have difficulties performing even simple arithmetical tasks, such as 4 + 5 (Molko et al. 2003). Also, the proficiency with which children and adults can solve nonverbal, approximate numerical tasks correlates strongly with their mathematical aptitude. Lourenco et al. (2012) found that college students who were better at estimating differences in number and cumulative area were better at advanced arithmetic and geometry.

Taken together, this evidence indicates that vertebrates and invertebrates can discriminate numerosities in their environment. Since animals spontaneously use numerical information to guide their decisions (*e.g.*, choosing a food source, approaching potential competitors, or joining a shoal), it seems plausible that numerical cognition has an evolved, adaptive function. Moreover, evolved numerical cognition also plays a critical role in our ability to engage in formal arithmetic. Thus, premise 1 seems fairly secure for the domain of number.

3.2. Functional properties of numerical cognition

To assess whether evolved numerical cognition supports realism or antirealism about numbers, we need to look at its functional properties.

A consensus (*e.g.*, Feigenson et al. 2004) holds that animals have two distinct systems for representing numerosities: one for small collections (the object-file system) and one for larger magnitudes (the magnitude system). This two-systems account explains why animals represent numerosities ≤ 3 precisely, and larger numbers only approximately. The object-file system (Fig. 1a) represents small (≤ 3) sets of discrete objects in a placeholder format as slots that are kept in working memory. For example, 2 entities are represented as follows: there is an entity, and there is another entity numerically distinct from it, and each entity is an object, and there is no other object:

$$(\exists x)(\exists y)\{(\text{object}[x] \;\&\; \text{object}[y]) \;\&\; x \neq y \;\&\; \forall z(\text{object}[z] \rightarrow [z = x] \vee [z = y])\}$$

Due to limitations on working memory, the object-file system is limited to 3 (sometimes 4, depending on individual variation). Animals lose track when having to represent an entity, another entity, another entity, yet another entity, etc. This explains why animals can discern collections of ≤ 3 at a glance, but are inaccurate for larger cardinalities (Feigenson & Carey 2005).

The approximate-magnitude system handles numerosities > 3 (Fig. 1b). There is disagreement about how this system works, so I will focus on one neural network model, the numerosity-detector model[2] (Dehaene & Changeux 1993; see also Dehaene 2007). In this model, numerosities are represented through a multi-layered neural network. *Perceptual input*, provided by visual, tactile, or auditory stimuli, constitutes the first layer of processing. The stimuli are converted into representations of discrete objects. For instance, our early visual processing detects boundaries between objects by their light and dark contrasts. These representations of discrete objects serve as input to the *location map*, the second layer of processing. The location map abstracts away from individual properties, converting each object into a separate, parallel location. The resulting output feeds into the *summation clusters*, neurons that fire at various thresholds (third layer). These summation clusters provide input to the *numerosity clusters*, which encode cardinal values (fourth layer). In rhesus monkeys, summation clusters have been located in the lateral intraparietal area (Roitman et al. 2007), whereas numerosity clusters have been found in the intraparietal sulci and prefrontal cortex (Nieder et al. 2006). Neurons in the *numerosity cluster* appear insensitive to the physical characteristics of objects (*e.g.*, their size or shape), but are sensitive to cardinality. They do not respond exclusively to given cardinalities but rather have a response that is distributed around given magnitudes. For example, in the rhesus monkey brain, individual neurons that have their optimal response rate when monkeys see 4 items also exhibit some activation for values between

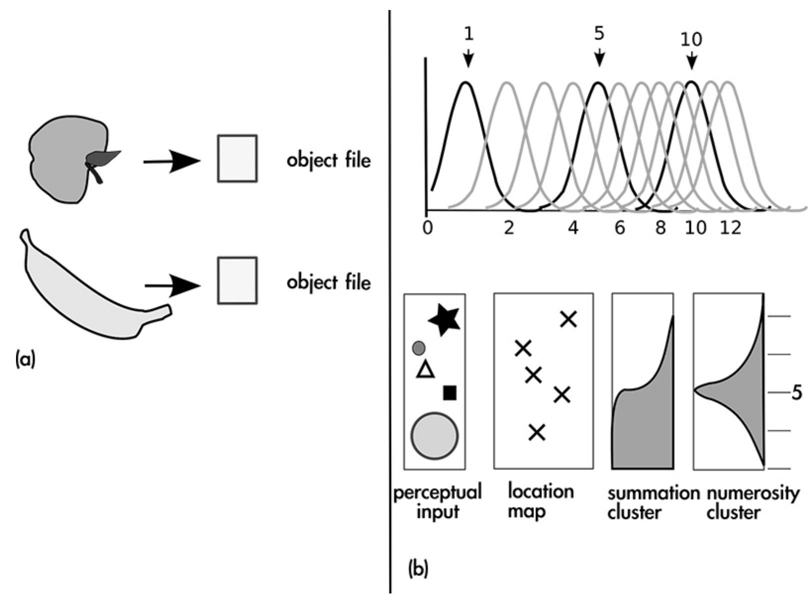

Figure 1: Two models of numerical cognition: (a) the object-file system for small numerosities ≤ 3, (b) the approximate-magnitude system for larger numerosities.

2. A competing model is the mode-control model (*e.g.*, Cordes et al. 2007), which proposes that numerical magnitudes are like cups of water that are being poured into a bucket (mental accumulator). Numerosities are represented along a mental number line, with each numerosity being represented by increasingly broadening tuning curves (scalar variability).

2 and 6 (Tudusciuc & Nieder 2007). Since neural resources are limited, this model predicts a logarithmic spacing of neural thresholds, such that a decreasing number of neurons are allocated to increasingly large numerosities. As a result, it becomes progressively harder to tell apart numerosities as they increase. Neurons in numerosity clusters respond to numerosities in a wide variety of formats, including visual, nonsymbolic, symbolic, and auditory formats (*e.g.*, Piazza et al. 2007).

4. A realist case for evolved numerical cognition

4.1 What does numerical cognition track?

Based on a thought experiment involving ancestors predicting the presence of lions behind bushes, Clarke-Doane (2012) argues that animals' adaptive responses would remain the same — assuming first-order logical truths remained constant — if the corresponding mathematical truths differed. Thus, even if 1 + 1 = 0 (a claim about numbers), we would still be better off believing 1 + 1 = 2 (our beliefs being in line with first-order logical truths about lions behind bushes), which provides the basis for premise 2 (Even if it were the case that ¬p in some realist sense, it would still have been more adaptively advantageous to believe that p).

One difficulty with evaluating the plausibility of this claim is that Clarke-Doane does not say anything about the psychological properties of the ancestors in this scenario. All he argues is that the first-order logical properties of the situation obviate any need to invoke numerical facts to understand the ancestors' behaviors. He holds that their behavior can be adequately understood with the first-order logical properties at hand, but is silent on how exactly this correspondence between first-order logical properties and adaptive behavior is achieved.[3]

Thanks to the wealth of empirical information about numerical cognition, we can examine whether Clarke-Doane's claim is plausible by probing how numerical cognition would work under the assumption of antirealism. How can animals adaptively respond to first-order logical properties in their environment by representing numerosities, given that truths about numbers do not co-vary with first-order logical truths? There is a candidate cognitive mechanism that would explain animal adaptive behavior in the absence of numbers: the object-file system. As we saw earlier, the object-file system (*e.g.*, Feigenson & Carey 2005) maps onto first-order logical properties of the environment. The object-file system would provide a straightforward explanation for how animals can represent numerosities, assuming antirealism about numbers. This would provide support for the antirealist case.

However, as we have seen, animals, infants, and even numerate adults who are prevented from counting cannot keep numerosities over 3 or 4 in working memory, and need to rely on approximate magnitudes rather than the object-file system. They cannot use object files when comparing or reckoning with numerosities > 3. When ancestors P and Q have to decide whether to forage at a bush with 50 fruits or one with 100 fruits, they need to use approximate magnitudes, rather than object files. Similarly, when ancestors P and Q evaluate whether to engage in a fight with ancestors from a nearby group, they rely on approximate magnitudes to examine whether their own group or the rival group is larger.

What do approximate magnitudes track? Given the naturalistic angle of evolutionary arguments for or against realism, it makes sense to let our ontological questions be informed by the scientific practices in a given domain (see Bangu 2012 for a defense of this claim). Scientific practice suggests a crucial explanatory role for numbers in research on numerical cognition. Cognitive scientists take care to isolate numerical properties (rather than other magnitude properties, such as visual density or continuous size) when testing animal numerical cognition. Such controls have become the standard in studies of mathematical cognition. For example, even when using naturalistic stimuli, such as fish in a shoal, to examine preferences for larger shoals, researchers take care to control for visual density and for total area of the shoals (Dadda et al. 2009). In their fMRI study of numerical cognition,

3. I am grateful to an anonymous referee for pressing me on this point.

Cantlon et al. (2006, 845–846) write, "Arrays consisted of blue circle elements that varied in density, cumulative surface area, spatial arrangement, and size, but were constant in both the number of elements (16 or 32) and in local element shape (circles). Thus, participants adapted to the constant number and shape of the elements." The authors did this to prevent neural adaptation to surface area, spatial arrangement, and size, as they wanted to exclusively focus on number and shape of the elements. After habituating participants, the authors found that the bilateral intraparietal sulci (IPS) in children and adults were more responsive to changes in number than changes in shape, suggesting that "the IPS, known to be part of a cerebral network important for symbolic number processing, is also recruited in nonsymbolic numerical processing" (Cantlon et al. 2006, 852). Neuroscientists also explicitly appeal to numbers to explain the function of numerosity clusters, which are insensitive to the physical characteristics of objects (*e.g.*, their size or shape), but respond to cardinality. Since the ancestors in the foraging and fight scenarios rely on the magnitude system, and the magnitude system represents numbers (at least according to cognitive scientists investigating it), we have prima facie support for the claim that animal mental representations of numerosities track numbers.

4.2 The indispensability of numbers for numerosities

I have so far argued that scientific practice provides a prima facie realist case for numbers, since neuroscientists and cognitive psychologists are interested in isolating numerical properties of the environment, and since they refer to numbers in their explanations. An antirealist might respond that although cognitive scientists who propose the magnitude system invoke numbers, they also use fictional entities such as location maps, and clearly there are no location maps in the brain. Scientists often use idealizations (such as frictionless slopes) that play a crucial role in their theories. A particular model can consist of real entities (*e.g.*, unobservables, such as electrons, and observables, such as results of measurements) as well as fictional entities (*e.g.*, computer simulations, idealizations). If a model is confirmed, this does not license belief in all its theoretical posits (Maddy 1992). It might be that mathematics plays an expressive role, an easy way to represent numerosities of objects that the brain represents, rather than a crucial explanatory role.

To make the positive case for realism, I propose that the best explanation for numerosities involves numbers — animals make representations of magnitude in the way they do because they are tracking structural (or other realist) properties of numbers. A number of mathematical realists (*e.g.*, Baker 2009, Lyon 2012) have formulated updated versions of the indispensability argument for mathematical realism, arguing that mathematical truths play an indispensable role in scientific explanations. If we are ontologically committed to the existence of unobservable scientific properties that play a crucial explanatory role (like electrons), we should also be ontologically committed to the existence of mathematical entities.

Baker (2005) considers the life cycles of species of the genus *Magicicada*. These North American insects have life cycles of either 13 or 17 years (depending on the species), consisting of a long nymphal stage underground followed by a brief adult phase of only a few weeks above ground. It is evolutionarily advantageous for magicicadas to have long life cycles that do not intersect with other cyclical periods: it helps them to avoid predators or matings with similar species. 13 and 17 are prime and thus do not intersect with smaller cyclical periods. The fact that 13 and 17 are prime is an essential element in the explanation of why the *Magicicada* life cycles have these particular durations. While there are various physical, nonmathematical factors in this explanation (*e.g.*, why it is adaptive for cicadas not to have life cycles that coincide with those of other species), the primeness of 13 and 17 is an essential element in the explanation. Thus, according to Baker (2009, 614), this case study provides an "indispensable, mathematical explanation of a purely physical phenomenon".

It seems mysterious that acausal entities can figure in causal explanations. Clearly, long prime cycles do not intersect with smaller cyclical periods, but it remains unclear how this mathematical fact could influence the evolutionary history of life cycles in *Magicicada*. Lyon (2012) draws on the distinction between process and program explanations (proposed by Jackson & Pettit 1990) to elucidate the role of acausal mathematical entities in scientific explanations. A process explanation provides a detailed account of the proximate causes of the event to be explained. A program explanation, by contrast, appeals to a property or entity that is not causally efficacious but that nevertheless ensures the instantiation of a causally efficacious property that is an actual cause. For example, a square peg does not fit into a round hole with a diameter equal to the side of the square. The geometric properties are causally relevant in the explanation, even though they are not causally efficacious. In this case, the process explanation appeals to physical properties (the impenetrability of the overlapping parts of the peg), whereas the program explanation cites geometric properties. The program explanation works at a higher level than the process explanation. While process explanations appeal to specific physical situations, program explanations provide modal information. They allow us to generalize: not only this particular peg and hole, but any square peg will not fit into any round hole where the diameter of the hole is equal to the side of the square. Similarly, the primeness of 13 and 17 provides a program explanation for the life cycles of *Magicicada*, even if these mathematical facts do not contribute to the actual physical processes that are involved.

Using this strategy for evolved numerical cognition, the realist can argue that numbers are indispensable for program explanations of numerosities. Consider the behavior of a lioness in McComb et al.'s (1994) tape recorder experiment. The lioness decides to approach the auditory signal of 3 roaring individuals when 7 members of her own pride are in the vicinity. A plausible explanation for her decision (one also presumed by the researchers who conducted these experiments) is that the lioness forms the belief $\text{numerosity}_{\text{own}} > \text{numerosity}_{\text{rival}}$. Physical factors in this explanation are the seven lionesses of the group and the voices of the three roaring intruders. In this case, the ratio difference between the numerosities of the groups is large enough to be discriminated by the approximate numerical system, which allows the lioness to rely on it. So the ratio differences between the numbers of lionesses heard on the tape recorder and the number of group members present (another mathematical fact) also has explanatory value. For instance, McComb et al. (1994) observed that when the tape recorder played 3 voices, the probability of any lioness approaching it was only 0.5 when 4 pride members were present, whereas it was close to 1 when 7 pride members were in the vicinity. The mathematical facts that $7 > 3$ and that the ratio difference between these numbers is relatively large provide a general, high-level explanation for the behavior of individual lionesses in this experiment, as the numerical composition of the own and rival groups guide their behavior.

To give another example where mathematical facts have explanatory value, take shoal selection in fish. Shoaling fish prefer to join large shoals to small ones to reduce their risk of being eaten: predators are confused by larger shoals, and there is safety in numbers, as a predator can eat only a limited number of individuals. Mosquito fish show a spontaneous preference for the larger of two groups, 3 versus 2 and 8 versus 4. To examine what guides their choice, Dadda et al. (2009) controlled for the visual density of different shoals, and for the total space a shoal occupied. Nevertheless, fish consistently chose the shoal composed of more individuals. They were even successful in gauging this when they were able to see only one fish at a time. Here too, mathematical facts, such as that $3 > 2$ and $8 > 4$, provide a parsimonious explanation both for features of numerical cognition (*e.g.*, choice of a larger shoal over a smaller one) and for the adaptive value of decisions based on numerical cues.

4.3 A realist account for numerosities

So far, I have suggested that what we know about evolved numerical cognition supports realism, drawing on the observation that scientific practice suggests a realist understanding of numbers, and that numbers are indispensable for explanations about mathematical cognition. Some realists, such as Joyce (2006) and Sinnott-Armstrong (2006), have argued that we can expect on evolutionary grounds that numerical beliefs track numerical facts, but have not explicated how this tracking is supposed to take place.

As with all naturalistic accounts of mathematics, the chief obstacle to fleshing out a functional account of numerical cognition from a realist perspective is the access problem (Benacerraf 1973). Benacerraf originally understood the access problem in terms of a causal theory of knowledge. More recent ways of dealing with this problem have moved away from this framing. For example, Field (1989, 26) glosses it as the challenge "to provide an account of the mechanisms that explain how our beliefs about these remote entities can so well reflect the facts about them" (see also Yap 2009). Still, any naturalistic account — causal or not — will have to grapple with the fact that the human mathematician is "a thoroughly natural being situated in the physical universe", and that therefore "any faculty that the knower has and can invoke in pursuit of knowledge must involve only natural processes amenable to ordinary scientific scrutiny" (Shapiro 1997, 110).

I will here focus on ante rem structuralism, to give a sense of how realism about numbers could be true, given what we know about mathematical cognition. Structuralism holds that mathematical theories describe structures and positions in them. According to Shapiro's (1997) ante rem structuralism, nonapplied mathematics is concerned with structures that are conceived of as abstract entities (platonic universals), *i.e.*, structures that exist independently and prior to any instantiations of them. The precise nature of these entities is left unspecified, as it is not essential to mathematical practice. Just as one can talk about a goalkeeper's function in soccer (*i.e.*, keeping the ball out of the goal) without going into detail about the precise properties of the person in this position (*e.g.*, hair color), a mathematician can talk about the natural number 2 as a position within the natural number structure without having to worry about which set-theoretical conceptualization captures 2 best. Mathematical structuralism is not concerned with the internal nature of mathematical objects (*e.g.*, numbers, functions), but with how they relate to each other. There is no restriction on the kinds of things that can exemplify relations (Shapiro 1997, chapter 3).

The access problem arises when numbers are conceived of as objects that can be considered in isolation, like cats or fridges. Thus conceived, knowledge of numbers seems dependent upon our ability to interact with them. In a platonist ontology, numbers are not spatiotemporal objects, so it is hard to conceive of an interaction between natural beings like us and numbers (Resnik 1981).

If numbers are positions in a certain structure, direct causal interaction is not required to acquire knowledge about them; it suffices to be familiarized with specific instances of the relevant structure. Shapiro (1997, chapter 4) develops an account of how humans learn to grasp patterns.[4] He outlines a rather elaborate staged model, where each stage is intended to account for our knowledge of increasingly complex mathematical structures. For evolved numerical cognition, only his stage one (abstraction) is relevant. Abstraction takes place when subjects learn to recognize patterns, such as cardinalities of small sets. They recognize that the 2-pattern is common to all systems that contain exactly two objects, the 3-pattern is common to systems with three objects, and so on. Shapiro invokes a domain-general capacity, termed *pattern recognition*, by which children learn the natural numbers:

> In part, our child starts to learn about cardinal structures by ostensive definition. The parent points to a group of four objects, says "four," then points to a different group of four objects and repeats the exercise. Eventually, the child learns to recognize the pattern itself (Shapiro 1997, 115).

This scenario provides a naturalistic account of how, from a realist point of view, children can learn about numbers. Unfortunately it fails to capture the actual cognitive processes involved. As we have seen, our brains come equipped with a set of domain-specific skills to

4. A similar account is Resnik's (1982, 97) "experiencing something as patterned". I will concentrate on Shapiro's account, as it is the more elaborate.

recognize numerosities, rather than with an undifferentiated capacity to recognize patterns. However, we can provide a structuralist developmental account that is compatible with the functional properties of numerical cognition, as follows. Even after extensive training, nonhuman animals (such as chimpanzees) fail to represent natural numbers > 3 precisely (Biro & Matsuzawa 2001). They represent numerosities approximately, with increasing imprecision with larger magnitudes. How then do humans learn to represent a natural number like 54? Through their object-file system, young children have an innate capacity to distinguish 1-patterns, 2-patterns, and 3-patterns. As they learn to count, they realize that these patterns correspond to the linguistic utterances 'one', 'two', and 'three'. Remarkably, preschoolers always learn the numbers 1, 2, and 3 in that order (*i. e.*, children learn that 'one' represents a unique cardinal value, then 'two', then only 'three'). While one would expect the next step is four, children make a crucial induction: they make an analogy between *next in the numeral list* and *next in series of object-files*: if n is followed by n + 1 in the counting sequence, adding an individual to a set with cardinal value n results in a set with cardinal value n + 1. Children generalize this to higher magnitudes, which helps them to understand the successor function (Sarnecka in press). Next to object-files, the approximate-magnitude system continues to play a critical role in arithmetical skills in adults, as it helps them to gain semantic access to symbolic representations of numerosities > 3. Cultural means, such as counting words, fingers and other body parts, and tallies, help to represent natural numbers (De Cruz 2008).

Why did natural selection not allow for animals to represent natural numbers > 3 exactly? This is probably due to the adaptive function of numerical cognition. For example, animals require increasingly large differences to distinguish between larger numbers. Smaller numbers are more ecologically relevant: the nutritional difference between 1 and 2 apples is large; the difference between 10 and 11 apples is marginal. A fish is a great deal safer in a shoal of 3 individuals than in one of 2, whereas the difference is negligible for shoals of 13 or 12 fish. There are also neural constraints: brains do not have the space to allocate clusters of neurons to numerosities of increasing size ad infinitum. The adaptive and neural constraints together explain why decreasing numbers of neurons are allocated to increasingly large numerosities. In this picture, arithmetical facts, realistically construed, form an indispensable part of a physical-cum-mathematical property complex.[5]

Ante rem structuralism is a realist ontology compatible with the evolved features of numerical cognition. It can provide a general and straightforward explanation of what animals detect in numerical cognition. It also meets a prima facie objection against mathematical realism, namely the access problem. While ante rem structuralism may not be the only way to connect numerical cognition and mathematical ontology, this is a first stab at providing such a connection, bolstering the case for mathematical realism.

5. Conclusion

There is a tension between realism about abstract objects and evolutionary accounts of human cognition. How can our evolved brains that only have access to the natural world acquire true beliefs about putative abstract entities like numbers and moral norms? Balaguer (1998, chapter 8) has argued that empirical evidence can never decide between realism and antirealism because we have no epistemic access to the acausal mathematical realm. Nevertheless, I have demonstrated that a closer look at evolved mathematical cognition — both its adaptive value and its functional properties — can address the evolutionary challenge to mathematical cognition. An influential version of this evolutionary challenge (Clarke-Doane 2012) does not provide any details of how numerical cognition is supposed to work under the

5. There are other forms of structuralism, such as modal structuralism (Hellman 1989). Modal structuralism holds that mathematical statements are statements about possible structures. Modal structuralists aren't ontologically committed to mathematical structures over and above the structures we perceive. They use S5 modal logic for this. Given that evolved numerical cognition is focused only on structures we perceive, it would require further philosophical work, beyond the scope of this paper, to determine which form of structuralism is most compatible with it.

assumption of antirealism. The properties of evolved numerical cognition are more readily explained by a realist ontology of numbers than by an antirealist one, and realism is also more in line with the practices of cognitive scientists who investigate animal and infant representations of number. I explored one realist view in detail, ante rem structuralism, to give a sense of how a realist, sophisticated understanding of numerical cognition could work.

My account does not provide a decisive argument for realism about numbers, but it poses a new challenge for the antirealist: to tell a plausible nominalist story that can explain the adaptive behaviors of animals that rely on numerosities, especially given the practices of cognitive scientists which suggest that numbers play a crucial explanatory role in animal adaptive behavior.

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