The development of children's relational thinking in intensive and comparative extensive quantity settings

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ABSTRACT

Children's daily lives are filled with examples of intensive quantities (e.g. speed, cost). However, over 30 years of research shows that the intensive quantity concept is difficult for children to grasp.

This thesis examines the factors that contribute to children's difficulties with relational thinking about intensive quantities. To achieve this, a series of studies were conducted in which children's performance on intensive quantity problems is compared with performance on comparable extensive quantity problems.

The first study (N=228) provided a general examination of the performance of children aged 4–9 years on different intensive quantity problems (direct relations, inverse relations, proportional equivalence and sampling). Its main contribution was to establish the sequence in the development of different aspects of children's relational thinking about intensive quantities. It also examined whether relational thinking success was sensitive to variations in problem presentation. Three types of problem presentation were used (Physical Demonstrations, Computer Diagrams, and Manipulatives) and little variation in children's performance was observed.

Studies 2–4, carried out on 7 to 9 year olds, examined how extensive and intensive quantity settings contribute to success with direct and inverse relations problems. Study 2 (N=113) and Study 3 (N=244) approached this question with a comparison methodology. In both studies problems were easier for children in extensive quantity settings. Study 3 also compared two forms of problem presentation. Children receiving problems with relational language (more, less, the same) performed significantly better than children receiving quantitative descriptions on inverse relations problems. Study 4 (N=121) extended the comparison of extensive and intensive relational thinking problems to computational problems. This comparison produced no significant differences in performance between extensive and intensive quantities.

Sufficient evidence was found with non-computational problems to argue that understanding intensive quantities as a quantity expressed as a ratio presents children with a unique challenge, which cannot be explained solely by the need to work with inverse relations. The educational implications of these findings and future directions for research are discussed.

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For my mum

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CHAPTER 1

1 INTRODUCTION

1.0 Aims

The aim of this thesis is to investigate how the type of quantity present in a problem – extensive or intensive – affects the development of children's understanding of multiplicative relations between quantities. A series of studies will examine children's ability to solve direct and inverse proportion problems presented to them in an extensive and an intensive quantity context.

The introduction is split into three sections; the first section considers the definition of the terms used in the thesis. The second section presents the main research questions of the thesis, which are considered in greater detail in the review of the literature, and the third section sets out the structure of the thesis.

1.1 Definition of Terms

The thesis makes comparisons between extensive and intensive quantities and it is therefore important to define these terms. There are significant differences and similarities between extensive and intensive quantities. Looking at the differences, Piaget (1941) made the distinction between extensive and intensive quantities explicit to developmental psychologists in his book on the child's conception of number. Here Piaget defines extensive quantities as:

"...the name given to any magnitude that is susceptible of actual addition, as for example mass or capacity – the mass of a body formed of two bodies is the sum of masses of the original bodies." (Piaget, 1941, p.244)

Extensive quantities are measured by the scalar value of a unit of the same type. For example, in the expression '5 cm', '5' is the number of centimetres (the scalar value) and the centimetres are units used to measure the length of an object. The basic logical principle involved in this measurement operation is the additivity of units: if two units are joined, the total is the sum of the parts.

Intensive quantities are defined by Piaget as:

"...the name given to any magnitude which is not susceptible of actual addition, as for example temperature. Two quantities of water at 15°C and 25°C respectively do not produce a mixture at 40°C." (Piaget, 1941, p.244)

Intensive quantities differ from extensive quantities in both of the previous characteristics: they are not measured by a unit of the same type and the principle of additivity does not apply. For example, speed – an intensive quantity – is measured by the ratio of distance over time, as in *miles per hour*. If a person travels for the first hour at the speed 80m/h and for the second hour at the speed of 60m/h, that person's speed for the total time travelled is not 140m/h. This type of problem referring to the non-additivity of intensive quantities is termed 'intensivity combination' (Stavy and Berkowitz, 1980), or 'joining of intensive quantities' (Desli, 1999).

A second major distinction between extensive and intensive quantities regards part/whole relations; with extensive quantities the parts are always equal to the whole. For example when pouring a specific volume of water from a jug into several glasses, the volume of water in each glass differs from the original volume held in the jug. Pouring the water back into the jug restores the whole (total volume of water). For intensive quantities, the part/whole relation is quite different. Taking the example of orange squash, the intensive quantity is the taste of the drink (the relation between orange concentrate and water). Pouring orange concentrate mixed with water into several glasses, regardless of the volume of squash within each glass, the intensive quantity (taste) remains unchanged. For intensive quantities, the parts are equal to the whole. Stavy and Berkowitz (1980) use the term 'intensivity division' to describe the unique part/whole relation of intensive quantities. It is important to clarify that 'intensivity division' is not applicable to all intensive quantities only those that involve explicit part/whole relations. Desli (1999) refers to this type of relation as 'sampling' to encapsulate the idea that taking a sample of an intensive quantity does not change it. The term 'sampling' is adopted throughout the remainder of the thesis when talking of Stavy and Berkowitz's 'intensivity division'.

Although differences exist between extensive and intensive quantities it is important to show that certain logical ideas apply to both. Strauss and Stavy (1982) present three logical ideas needed for the concept of intensive quantity, which also occur with extensive quantities:

"The first way is the direct function ... an increase in the numerator increases the ratio. The second way is the inverse function, whereas an increase in the denominator decreases the ratio. The third way is proportions were both the numerator and denominator vary equally." (Strauss and Stavy, 1982, p.551)

These ideas can be illustrated within the context of extensive and intensive quantities, starting with relations between three extensive quantities (number of pigs, amount of pig food, and number of days). If two farmers have the same number of pigs (quantity 1) but different quantities of pig food (quantity 2), then assuming both farmers feed their pigs the same amount of food each day, the farmer with a larger quantity of food will be able to feed his pigs for more days than the other farmer (quantity 3). This is the direct function. If the number of pigs varies while both farmers have the same quantity of pig food, the farmer with fewer pigs will be able to feed his pigs for more days than the other farmer. This is the inverse function. And finally to illustrate proportional relations; if the first farmer has twice as many pigs and twice as much food as the second farmer, they will be able to feed their pigs for the same number of days. These relations with an intensive quantity can be illustrated in another way; if two drinks contain the same volume of water, increasing the volume of orange squash in the first drink will increase the relative sweetness of that drink. This is the direct function. Alternatively, starting with the same volume of orange squash and increasing the volume of water added to the first drink invokes the inverse function reducing the relative sweetness of the first drink. Finally, to make a larger volume of orange squash with the same taste as a smaller volume then the volume of both the orange squash and water must increase proportionally.

Intensive quantities of this type are encountered in everyday life by adults and children, at school, in the workplace and in science. Buying food in the supermarket involves finding the best deals, while nurses in hospitals have to work with intensive quantities when administering medication to patients. In science, concepts involving extensive quantities such as temperature, density, and speed have been shown to be difficult for children to work with (Strauss and Stavy, 1982; Singer, Kohn, and Resnick, 1997; Piaget 1970). As anything measured in ratios is an intensive quantity, it becomes very important that students grasp this concept.

The question remains as to whether the distinction between extensive and intensive quantities is important enough to be made explicit to children.

If we refer to the current educational practice of British primary schools, the answer must be a negative. There is no explicit mention of intensive quantities per se in the national numeracy strategy (Department for Education and Employment [DfES], 1999), though the idea of cost is mentioned explicitly as a term children should understand from Year 3 (7–8 years) onwards. This educational policy contrasts with the current practice in Japanese education, where by the end of primary schooling, children will have received ten lessons focused on developing their conceptual understanding and computational skill with intensive or 'per unit' quantities (Ishida, 2004).

1.2 Background

1.2.1 The Distinction Between Additive and Multiplicative Reasoning

When talking about the distinction between additive and multiplicative reasoning it is important to separate procedural from conceptual understanding. Procedurally strong connections appear between addition and multiplication in the teaching of multiplication as repeated addition. It is perfectly correct to solve the problem 24 × 3 as 24 + 24 + 24. Indeed researchers Fischbein, Deri, Nello, and Marino (1985) suggest that repeated addition is the implicit model of multiplication held by children. Yet later work by Nesher (1988) shows that the addition technique is a reflection of teaching methods rather than an implicit model. If repeated addition represents the often-taught procedural similarity of addition and multiplication, what then are the conceptual differences? Piaget's work (1941) on the child's conception of number highlights the unique conceptual challenge of multiplicative reasoning. Piaget shows that one-to-many-correspondence marks the beginning of a conceptual understanding of multiplication.

Piaget tested the understanding of one-to-many-correspondence by asking children to put pairs of flowers (blue and pink) into vases. Piaget then hid the flowers and asked the children to place next to each vase the right number of tubes (each able to hold one flower) as they would need for all the hidden flowers. If children understood one-to-many-correspondence, they would see that each vase corresponds to two flowers.

Piaget classified children into three developmental stages. Children at Stage 1 could not make the one-to-many-correspondence:

"...these children merely make an arbitrary estimate of the increase and are unaware of the duplication. Although they understand that n blue flowers correspond to n vases, and that n pink flowers also correspond to the same vases, they do not understand that the n vases correspond to pairs." (Piaget, 1941, p.215)

Children at Stage 2 initially lined up tubes and vases in one-to-one-correspondence, though realising their error create a second row of tubes. During Stage 3 (around 6–7 years) children apply one-to-many-correspondence spontaneously.

Nunes and Bryant (1996) explain why one-to-many-correspondence represents an important distinction between additive and multiplicative reasoning. Firstly, one-to-many-correspondence involves maintaining a constant relation between two sets leading to the concept of ratio. This contrasts with additive relations where maintaining a constant difference involves adding the same number of objects to each set. Secondly, maintaining a one-to-many-correspondence requires replicating the ratio (two flowers for one vase), which is unlike additive reasoning where any number can be added. Thirdly, even though the total number of vases and flowers can increase the ratio remains unchanged. And finally, one-to-many-correspondence leads to a new type of number called the 'scalar' (p.146), the scalar refers to the number of replications carried out. For example, in this case, the scalar value of 10 times would refer not to the quantity of vases or flowers, but to the required number of vase/flower replications.

Although Piaget's early work established that young children understand some aspects of multiplication, the mastery of multiplicative reasoning can take many years. Piaget's later experiments on the schema of proportionality (Piaget and Inhelder, 1958, 1975; Piaget, Grize, Szeminska, and Bang, 1977) looked at why the conceptual mastery of multiplication is quite protracted. These experiments tend to fall into two broad categories. In the first category, Piaget used non-computational experiments to study the development of logical multiplication via separation of variables. In the second category he used computational experiments looking at the progression from logical multiplication to true proportional reasoning. Together these two categories provide clues as to the development of the concept of multiplication.

In the experiments exploring the separation of variables, Piaget investigated how children come to understand that establishing causal relations is only possible after other potential causes are controlled. For example, the flexibility of a metal rod, for a given metal, is an intensive quantity directly proportional to length and inversely proportional to thickness. Inhelder and Piaget (1958) report that children younger than 11 do not understand that the systematic control of variables is important to test the effect of varying quantities on the flexibility of metal rods. Intensive quantities might be difficult to understand as knowledge of the direct or inverse proportional relations requires clear understanding that all other quantities must be held constant. Indeed Piaget's (Inhelder and Piaget, 1958) second wave of experiments on the computation of proportional relations takes this idea further. The results suggest that when children are able to choose between working with direct or inversely proportional relations they prefer to work with direct relations. This result in turn suggests that inverse relations may account for some of the difficulty children have understanding intensive quantities.

A second process essential to the understanding of intensive quantities is quantification. Extensive quantities are represented by a single integer. Presenting an array of objects to a child is all the child needs to proceed to quantification. As intensive quantities always result from a relation between two variables, the relation between variables is multiplicative and not additive. Piaget, Grize, Szeminska, and Bang (1977) showed that this is a difficult distinction for children to make. Children before the age of 11 years display a tendency to use additive reasoning incorrectly to solve proportional reasoning problems. This misconception involves adding quantities in order to maintain a constant difference rather than replicating an established ratio between quantities.

Piaget's body of work has proved important in the field of children's mathematical competencies. It highlights the clear distinction between additive and multiplicative reasoning while also showing that children's progression from additive to multiplicative reasoning is a long and difficult one. Piaget's writings on the distinction between additive and multiplicative reasoning, and his descriptions of children's tendency to provide additive solutions to multiplicative problems, are points not seriously disputed by anyone. Indeed, later investigations into secondary school students' proportional reasoning by Hart (1981) and more recently by Misailidou and Williams (2003) have shown that even between the ages of 11 and 15, students still present additive solutions to proportional reasoning problems.

Piaget explored the underlying mechanisms of children's thinking using a variety of experiments to highlight common schemas. A complementary approach to studying children's mathematical development undertaken by psychologists looks at how the underlying structure of problems influences children's reasoning. The classification of additive reasoning problems by Vergnaud (1982) in this area are widely accepted. The best way in which to classify multiplicative problems though remains a source of controversy in need of clarification.

1.2.2 The Classification of Multiplicative Problems

Since the early 1980s, attempts have been made in the psychology of mathematics education literature to develop a coherent system of classifying types of multiplication problems. The goal of this debate was to establish why some types of multiplication problem are more difficult than others and to suggest which type of problems could help improve children's awareness of multiplicative relations. The two theories to emerge from this debate were Vergnaud's (1983) theory of multiplicative structures and Schwartz's (1988) theory of referent transforming operations.

1.2.2.1 Vergnaud's Theory of Multiplicative Structures

Vergnaud's (1983, 1988, 1994) main contention is that even the simplest multiplication problems involving magnitudes are connected by four related numbers. It is from these four-term relations that children must extract three-term relations. Four-term relations are important because the multiplication of magnitudes involves two or more measures connected to one another by specific ratios. Viewing multiplicative problems as four-term relations led Vergnaud to create a classification system within which three distinct types of multiplicative structures exist: i) isomorphism of measures, ii) product of measures and iii) multiple proportions.

'Isomorphism of measures' is the simplest of the three structures according to Vergnaud (1983):

"The isomorphism of measures is a structure that consists of a simple direct proportion between two measure spaces M1 and M2." (p.129)

Isomorphism of measures problems often appear to involve only three terms, as a fourth term, which is always one, is often implicit. To illustrate this Vergnaud gives the example of the relation between two measures: cakes (M1) and money (M2). If four cakes are priced at five pence each, how much does the person have to pay? A representation of the problem can be been in Figure 1.1.

M1	M2
1	5
4	?

Figure 1.1 Vergnaud's representation of an isomorphism of measures structure

For this problem, 'five pence each' refers to the implicit ratio of five pence to one cake. Viewing this as a four-term problem leads to different possible solutions. Those of most interest to the current thesis were termed scalar and functional by Vergnaud (Figure 1.2).

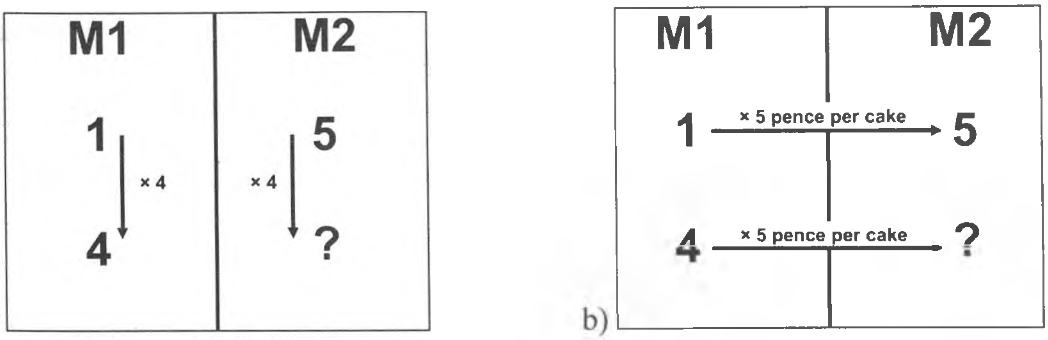


Figure 1.2 Scalar (a) and functional (b) solutions to Vergnaud's cake problem

Use of scalar solutions involves reasoning that there are four times as many cakes sold, so the person must pay four times the amount of money (5×4). Figure 1.2(a) shows how the scalar method establishes a ratio within the known measure, and transfers this onto the measure with the unknown quantity to reach the solution.

This contrasts with the functional solution (Figure 1.2(b)) where the constant coefficient between measures (cakes (M1) and money (M2)) provides the solution. This involves reasoning that if one cake costs five pence then multiplying the number of cakes required by the cost of one cake will lead to the total cost of four cakes (4×5). As can be seen in Figure 1.2, in both types of solution the implicit fourth term of one is important to the solution.

Vergnaud (1983) explains that children prefer to work with scalar solutions that mirror closely a simpler additive structure. For example, the scalar solution – four times as many cakes means four times as many 5ps – is conceptually similar to repeated addition – 5p + 5p + 5p + 5p. Working within measure spaces makes sense, as every increase in the number of cakes leads to an equivalent increase in the number of 5ps. Although it is possible to represent the functional solution (5×4) as repeated addition, it makes no sense in terms of the quantities in the problem. Adding 4 cakes + 4 cakes + 4 cakes + 4 cakes + 4 cakes = 20p although mathematically accurate has no real world meaning. For this reason, Vergnaud points out that adding cakes to reach pence is a strategy children do not employ. Other researchers using different terminology have also made this distinction between scalar and functional reasoning. Freudenthal (1978) for example uses the term 'internal ratio' (p.293) when talking about scalar solutions and 'external ratio' (p.293) when referring to functional solutions. Another popular choice of terms coined by Noelting (1980) refers to scalar solutions as the 'between strategies' of children (p.334) and functional solutions as 'within strategies' (p.334). Although these terms are adopted by different researchers in the subsequent literature review, they refer to the same ideas.

Although the isomorphism of measures structure covers many of the multiplication problems faced by children, multiplicative structures become complex when situations involve more than two magnitudes. For these situations, Vergnaud (1983) created the distinction between 'product of measures' and 'multiple proportion' situations.

According to Vergnaud :

"The product of measures is a structure that consists of the Cartesian composition of two measure spaces, M1 and M2 into a third, M3." (p.134)

This third quantity according to Vergnaud is always directly proportional to the other two. A typical Cartesian product would involve the multiplication of a set number of shirts (M1) by a set number of shorts (M2) to produce a number of possible outfits (M3). The product of shirts and shorts (outfits) represents a new quantity that is neither shirts nor shorts, but a measure of the total number of possible shirt/short combinations.

An important point made by Vergnaud regarding product of measures problems is that M3 (the product) remains an elementary quantity. That is a quantity still susceptible to addition (1 outfit + 1 outfit = 2 outfits). Vergnaud contrasts this to an isomorphism of measures structure involving distance, time and constant speed. He points out although the product of measures structure seems applicable to the isomorphism of measures structure (speed × time = distance), speed is a constant and not a variable; it is also a derived magnitude representing a distance/time quotient and is therefore not a true product of measures situation.

Vergnaud's third and final category of multiplicative structures, 'multiple proportions', is similar to product of measures problems in the sense that in their most basic form they involve relations between three elementary quantities. When we consider multiple proportions problems, we find less strict rules apply than for product of measures problems. With product of measures problems, a fixed pattern of quantity selection operates with the quantity that becomes the product retaining some property of the referent quantities. For example, cm × cm results in cm^2, or shirts × shorts results in outfits. On the other hand, in multiple proportions problems, this is not generally the case. As Vergnaud (1983) points out:

"In multiple proportions, the magnitudes involved have their own intrinsic meaning, and none of them can be reduced to a product of the others." (p.139)

On page 3, during the discussion of direct and inverse proportional relations with extensive quantities, I presented an example of a multiple proportions problem. A relation was established between three extensive quantities: pigs, food, and days. Thinking about the way in which the consumption of a fixed quantity of food by a certain number of pigs takes a certain number of days creates relations between three previously unrelated quantities. Each quantity within the multiple proportions structure –pigs, food and days – though related within the problem setting, remain independent magnitudes.

1.2.2.2 Schwartz's Theory of Referent Transforming Operations

A different approach to the classification of multiplicative problems is presented by Schwartz (1988), who expresses the classification of multiplication problems with his theory of referent transforming operations. Schwartz promotes multiplicative structures as three-term relations rather than the four-term relations put forward by Vergnaud (1983).

The three-term approach stems from Schwartz's argument that the link between numbers and referents underpins important conceptual distinctions between multiplicative and additive structures. Viewing multiplication situations in terms of referents requires a distinction between extensive and intensive quantities that is not made by Vergnaud (1983).

When considering the example in 1.2.2.1, Schwartz's approach would term the number of cakes to be bought and the unknown total cost as extensive quantities, and the cost of one cake as an intensive quantity. Cake and money totals are both extensive quantities because they are elementary quantities with their own intrinsic meaning. The extensive quantities in the cost of cakes problem describe quantity totals while saying very little about the relation that exists between these quantities. The intensive quantity (price per cake) provides a different type of description of the relation between money and cakes. This new 'intensive quantity' is a descriptor of the quality of the cakes and relates to Vergnaud's use of derived quantity. The intensive quantity of cost per cake remains unchanged regardless of whether the purchase involves one cake or a whole box of cakes.

Like Vergnaud, Schwartz's theory of referent transforming operations leads to three types of multiplicative structures. The majority of situations faced by children involve the triad of one intensive and two extensive quantities (I E E') as illustrated by the cost of cakes problem. This structure incorporates many of the problems Vergnaud regarded as isomorphism of measures. Schwartz also distinguished between a triad of extensive quantities (E E' E') which corresponds closely to Vergnaud's product of measures situations and a triad of intensive quantities (I I' I') that could be regarded as complex multiple proportions situations. It is the distinction between (I E E') and the isomorphism of measures classification that is of the most interest, as it highlights why Vergnaud (1983) and Schwartz (1988) adopted different methods of classification.

The link between Vergnaud's approach to the classification of multiplicative structures and his earlier work (Vergnaud, 1982) on the classification of additive structures influences the four-term classification. Using a four-term approach allowed Vergnaud to make connections between children's development from additive reasoning problems to multiplicative reasoning problems. It also helped to make clear why children favour scalar relations over functional relations. However, by adopting this, Vergnaud did not give intensive quantities a role in his theory. This is primarily due to Vergnaud's failure to consider inverse relations as something that distinguishes intensive quantities from product of measures problems. This led Vergnaud (1983) when discussing Noelting's (1980) experiments on taste of orange squash (an intensive quantity) to view taste as a multiple proportions problems rather than a special kind of product of measures problem. He indicates that in these problems there are two variables, amount of concentrate and amount of water, in direct proportion to a third one, taste. This is incorrect as taste is actually a product of the first two, one directly proportional and the other inversely proportional to how orangey the juice tastes.

When Schwartz (1988) developed his system of classification, he had different priorities to Vergnaud. For Schwartz there was the need to develop a system that made explicit the distinction between multiplicative and additive structures.

According to Schwartz (1988) the majority of problems encountered by children are of the I E E' structure and should be introduced as such for children to think about the relations between referents. Viewing multiplication as a four-term relation according to Schwartz would be a mistake, as it would help reinforce the conceptual misunderstanding of scalar reasoning that multiplication always leads to bigger numbers. Taking Schwartz's view of the coefficient of price per cake as an intensive quantity demands a deeper understanding of the relation between quantities and their referents. The relation of cost to money and product can involve both directly and inversely proportional relations. For example, with cost controlling the amount of product purchased, increasing the amount of money paid for the product leads to an increase in the product's cost – the direct relation between money and cost. If money is controlled and the size of the product purchased varies, then an increase in product size leads to a decrease in the product's cost. This result is the inverse relation between product size and cost. Thinking about the inverse relation between quantities shows why children must be able to think beyond the scalar approach to multiplication problems.

1.2.2.3 Resolving the Vergnaud vs. Schwartz Debate

Examining both Vergnaud's and Schwartz's theories leads on to the question of whether the differences between them can be resolved. Both theories suggest different ways of classifying multiplication problems, but which of them captures the important distinctions between multiplicative reasoning problems most accurately? The main tension between Vergnaud and Schwartz is whether intensive quantities have a role to play in the classification of multiplicative problems. This in turn leads to the question of whether Vergnaud (1983) was wrong to disregard the role of inversely proportional relations as a significant multiplicative structure. Finally, having considered these questions, is it possible to make the case that Schwartz's (1988) distinction between extensive and intensive quantity problems should be synthesised with Vergnaud's theory to arrive at a more complete theory of classifying multiplication problems?

If Vergnaud (1983) is correct and problems involving the comparison of intensive quantities are the same as general multiple proportions problems involving three independent (or extensive quantities) then the distinction between extensive and intensive quantities should not determine the relative difficulty of problems.

If Schwartz's (1988) idea about the importance of referents is correct then comparing extensive with intensive quantities will show differences between the relative difficulties of problems. This would indicate that Vergnaud's theory requires revising before being considered the definitive theory of multiplicative structures. As Schwartz never tested the distinction between extensive and intensive quantities, my thesis will address this conflict.

1.3 Problems Investigated in the Thesis

The thesis focuses on children's ability to solve intensive quantity direct and inverse relations problems. Direct and inverse relations are studied as they represent the first significant logical gains children make with intensive quantities (Piaget and Inhelder, 1974; Noelting, 1980; Strauss and Stavy, 1982). Studying the beginnings of children's ability to work with intensive quantities will allow the examination of methodological and conceptual issues in children's development.

The methodological question addressed in the thesis is how children's ability to think about direct and inverse relations is influenced by the manipulation of problem presentation. The types of contrasts made are drawn from significant results in the wider proportional reasoning literature such as the use of manipulatives (Tourniaire and Pulos, 1985; Misailidou and Williams, 2003), realistic and diagrammatical forms of presentation (Schwartz and Moore, 1998), and the use of relational and quantitative information (Harel, Behr, Lesh and Post, 1994; Cowan and Sutcliffe, 1991; Desli, 1999).

There are two conceptual issues addressed in the thesis. The first issue is the conceptual difficulty of differentiating the different logical relations within the intensive quantity concept. This is achieved by revisiting the research on the development in children of the logical relations necessary for understanding sweetness (Strauss and Stavy, 1982). The current thesis widens the scope of Strauss and Stavy's analysis to ask whether the same order of problem difficulty is observed in the intensive quantity setting of taste.

The second conceptual issue is whether intensive quantity should be treated as a multiplicative structure which offers unique conceptual challenges for children or whether, as Vergnaud (1983) suggests, the intensive quantity is an example of a multiple proportions problem. The question here is whether children's initial difficulties with intensive quantity relational thinking problems can be explained by a general difficulty of understanding inverse relation common to all multiple proportions problems. If children's relational thinking with intensive quantities is only about understanding inverse relations then it should follow that problems presented in extensive and intensive quantity settings should produce similar levels of difficulty.

1.4 Structure of the Thesis

The review of the literature regarding previous research from the proportional reasoning literature carried out on children's reasoning in extensive or intensive quantity contexts is presented in Chapter 2. This review focuses on how the distinction between extensive and intensive quantities is handled in the current literature. Also of interest is how the distinctions between direct and inverse relations problems, which are important in Schwartz's but not in Vergnaud's theory are handled in the literature.

Chapter 3 presents the first empirical study of the thesis. The first study consists of two experiments and looks specifically at the concept of intensive quantity that Vergnaud omitted from his theory. Study 1 addresses two issues; Experiment 1 considers how the mode of presentation affects children's reasoning about direct and inverse relations with intensive quantities. Comparison is made between three modes of presentation which the literature suggests influence children's thinking about proportional relations – physical demonstrations, computer diagrams and problems presented with manipulatives. Experiment 2 returns to a study conducted by Strauss and Stavy (1982) which looked at the development of logical relations required to understand sweetness. This experiment examines whether the developmental pathway identified by Strauss and Stavy (1982) can be extended to the intensive quantity of taste.

The second study is presented in Chapter 4. Study 2 offers the first systematic comparison of intensive and extensive quantities problems involving direct and inverse relations. If Vergnaud's current theory is adequate then these comparisons should provide no significant differences between the performance of children on extensive and intensive quantity problems.

Study 3 is presented in Chapter 5. This study attempts to replicate the findings of Study 2 while varying the presence of quantitative descriptions of the quantities in the problem in order to judge whether this affects children's judgements. The children are split into two groups: Group 1 receives problems with quantitative descriptions of the quantities, while Group 2 receives the same problems with relational descriptions of the quantities (more, less, same).

Chapter 6 presents Study 4 which examines how children's ability to perform calculations with quantities is affected by the presence of extensive and intensive quantities within the problem. The impact of missing value problems on children's ability to work with direct and inverse relations is also addressed in this chapter.

The final chapter, Chapter 7, will present the general and overall discussion of the results and the conclusions that can be drawn from the reported studies and the published literature. Consideration will also be made of the limitations of the research done for this thesis, and suggestions for possible future research that would build on the results of this thesis.

CHAPTER 2

2 THE REVIEW OF THE LITERATURE

The aim of the thesis is to investigate how type of quantity – extensive or intensive – affects children's developing understanding of multiplicative relations between quantities; particularly whether intensive quantities represent a unique type of problem of which children have to become aware. The thesis studies in detail the origins of the intensive quantity concept by examining children's relational thinking about intensive quantities. Therefore, the review of the literature is primarily concerned with the proportional reasoning of children aged 5–11 years as this age range represents a central period in the development of relational thinking.

As intensive quantities are observed in many children's day-to-day interactions with quantities it must be clarified that the current thesis considers the logical development of relational thinking to be quite independent from perceptual awareness. For example, speed is an intensive quantity that many children study and come into contact with on a daily basis, whether it is being driven to school in a car, riding a bicycle, or running around the playground with friends. When experiencing speed, children develop a perceptual awareness of changes in speed through the perception of motion. This is different to the logical understanding that speed is dependent on the relation between the previously unrelated dimensions of distance and time. The logical understanding of an intensive quantity initially requires the understanding of two concepts. The first concept required is that an intensive quantity is composed of two quantities from different dimensions. The second concept is that these referent (or extensive quantities) can be both directly and inversely related to the intensive quantity they form. Intensive quantities always involve one extensive quantity that is directly proportional to the intensive quantity and another that is inversely proportional to it. In other words, intensive quantities are measured by one quantity divided by another. Speed is the result of the relation between two previously unrelated quantities (distance and time). Therefore, if the time travelled stays the same, and the distance increases, then the speed increases, this is the direct relation; whereas if the distance stays the same, and the time increases, then the speed decreases, this is the inverse relation. Speed is measured in miles (distance) per hour (time).

2.0.1 Selection of Literature

The following literature review only considers the development of direct and inverse relational thinking of children between the ages of 5 and 11 in a variety of intensive and extensive quantity contexts. It must be noted that two specific areas of research within the targeted literature are excluded.

Firstly, this thesis only deals with intensive quantities and relational thinking problems represented using the language of ratio. It is also possible to represent intensive quantity problems using the language of fractions, provided the problem involves explicit part/whole relations. For example, the taste of a drink of orange squash can be described using the language of ratios (1 measure of squash to 2 measures of water) or the language of fractions (⅓ squash to ⅔ water). However, as it is not possible to use fractional language to represent all intensive quantity problems (e.g. speed, cost) the issue of linguistic representation is not considered in this thesis. The question of representing intensive quantities using either the language of fractions or ratios has though been considered elsewhere; see Howe, Nunes and Bryant (2004).

Also excluded from the current literature review is work on intensive quantities composed of non-perceivable quantities, which lack meaning within the child's culture. The most notable intensive quantity to be excluded from the literature review on these grounds is temperature. It could be argued that temperature is an intensive quantity with which children are familiar, if only perceptually. It is also an area with an established body of research (Erickson, 1979, 1985; Shayer and Wylam, 1981; Cowan and Sutcliffe, 1991).

However, in terms of the problem currently under investigation – the coordination of relations between extensive quantities – it is a difficult quantity. Temperature involves non-perceivable quantities such as 'kinetic energy' which have little or no meaning for the children participating in the research for this thesis. This means that understanding temperature in terms of coordinating relations is beset with an additional requirement of specialised knowledge. Therefore using an intensive quantity like temperature to study relational thinking would make it difficult to draw conclusions about the extent of children's relational knowledge. As a result, this body of research is not considered in the literature review or the wider thesis. The one exception is density which can be thought of in terms of a mass to volume ratio which is difficult for children to conceive of (Fassoulopoulos, Kariotoglou, and Koumaras, 2003) but can be represented more simply in terms of crowdedness.

2.0.2 Organisation of the Literature Review

Chapter 2 is organised in three sections. Section 1 presents evidence from Piaget's studies on the origins of proportional reasoning. Piaget's work serves as a suitable starting point as many of his studies consider the conflict between perceptual awareness and the underlying logic of the problem. This review of Piaget's work follows his interest in the development of logic over perception and starts with Piaget's first studies of children's coordination of relations between different dimensions, through to his later studies on the proportionality schema which considered the role of direct and inversely proportional relations. His work is also a suitable starting point for the review of the literature as many of the ideas prevalent in the proportional reasoning literature today can be traced back to his experiments.

Section 2 deals with post-Piagetian research into proportional reasoning with primary-school age children. Attempts to work within both the Piagetian paradigm of logical schemas and the alternative approach of problem classification are considered. The case is made that in addition to the study of logical relations identified by Piaget, it is also important to consider the underlying mathematical structure of the problem when attempting to understand the difficulties children have with relational thinking about intensive quantities. Evidence supporting the distinctions of the classification theories of Vergnaud (1983, 1988) and Schwartz (1988) is presented to argue for the necessity of systematic comparison of relational thinking in intensive and extensive quantity contexts.

Section 3 describes the current learning opportunities children aged between 5 and 10 years have with intensive quantities in the English primary school system. The aim of this section is to establish what can be said about children's familiarity with intensive quantities prior to the studies conducted for the current thesis. As already stated, intensive quantities are present in the day-to-day activities of many children as perceptual entities. In order to try to understand more clearly the progress children make in their logical reasoning about intensive quantities, it is important to establish what children are taught at school. As Nunes and Bryant (1996) point out, a familiar context does not automatically lead to an understanding of the logic of the situation:

"when making orange juice, (children) are probably happy to add water slowly and taste the juice until the flavour is just right, rather than to think about mathematical proportions. Although the situation is certainly familiar, it is not a familiar mathematical situation." (p.170)

To establish a child's learning opportunities in the current English primary curriculum, the third section considers the steps taken by the original and updated numeracy strategies (DfEE 1999, 2003) to introduce children to the intensive quantity concept.

2.1 Section 1: The Piagetian Approach to Proportional Reasoning

This selection of Piaget's work is used to show how many distinctions used in the developmental psychology literature today were created by Piaget. The Piagetian experiments covered here are drawn from various periods of Piaget's writing and are used to present two broad themes from his work. The first theme regards the origins of thinking about relations between quantities. It will be shown across many experiments that the initial problem for children is to consider independent dimensions simultaneously. This is true whether both dimensions were directly related to the quantity as with the conservation experiment (Piaget, 1941), or whether direct and inverse relations are involved, as with the projection of shadows and speed experiments (Inhelder and Piaget, 1958; Piaget, 1970). It will also be shown through Piaget's experiments that consistently children's first success with relations between quantities involves seriation problems – when one dimension is controlled and the second varies. When dealing with these problems, children begin to think of the varied dimension as directly or inversely related to the outcome quantity. This development is an important prerequisite to the understanding of part/whole relations between quantities.

The second theme presented from Piaget's work is methodological and results from his insistence that knowledge does not come from perception. It is clear in each experiment that relying on perception never leads to systematic success and will often result in children producing contradictory arguments. This lesson is important for the current thesis as it will show that to understand children's logical development, problems must be designed that cannot be solved through perception alone.

This section starts with one of Piaget's earliest studies, the conservation of continuous quantities. He was the first researcher to consider the logical schemas needed to work with relations between quantities and how are they acquired by children.

2.1.1 The Conservation of Continuous Quantities (1941)

During his conservation of continuous quantities experiment, Piaget (1941) asked children to compare the volume of water in two glasses. Volume is a relation between quantities problem dependent on the relation between three independent quantities (height, width and depth). Piaget simplified the problems, asking children to coordinate only two dimensions (height and cross-section). To look at how logical awareness of volume as a relation between quantities problem develops; Piaget designed problems in which there was a divergence between perceptual information and the correct solution. Piaget's divergence problems took three forms: in the first a glass (A) was filled with water and the child was given an empty glass (L), which was both taller and narrower than glass A. The child was then asked to pour the same volume of water into glass L as was seen in glass A.

This first problem was not actually about conservation, but about whether children could pay attention to two quantities simultaneously (height and cross-section). Piaget referred to this logical requirement as the schema of logical multiplication and regarded it as an important prerequisite to both the conservation and wider proportionality schemas.

The second form of divergence problem involved the classic test of children's ability to conserve volume. Two glasses of equal size (A_1 and A_2) were filled with water, and then the water from glass A_1 was poured into glass L. The child had to decide whether the volumes of water in A_2 and L remained the same. This problem requires the quantification of changes of height and cross-section to understand that though these quantities vary in opposite directions between glasses (A and L), these differences compensate one another. Piaget refers to this as an inversely proportional relation (p.13). It must be made clear that this is different to the idea of inverse relations studied during the current thesis. The idea of inverse relations studied during the conservation experiment is more accurately described as an inverse compensation problem. This is because logical multiplication in volume problems refers to the idea that to maintain a constant volume, any increase in the height of a fluid must be compensated for by an equivalent decrease in the cross-section. This form of inverse compensation is only possible because both height and cross-section are directly proportional to volume. This is different from the notion of inverse relations relating to taste, for example, controlling the quantity of sugar, and varying the quantity of water invokes the inverse relation of less water leading to a sweeter taste.

The third form of divergence problem involved glasses A_1 and A_2 with A_1 this time poured into a series of smaller glasses, i.e. B_1 and B_2. Here the problem is of conservation in the context of the part/whole relations of extensive quantities, in order to explore whether children understand that the whole is equal to the sum of its parts.

Piaget reports that children's responses to these three problems could be classified into three developmental stages. During the first stage, children had little notion of volume as a relations between quantities problem and considered only the appearance of the liquid, equating perceivable differences to changes in volume. According to Piaget, children's perceptual misconceptions occur because they focus on only one dimension at a time without coordinating dimensions. For Piaget the dimension on which children focused on was largely irrelevant; rather it was the lack of coordination that was significant:

"The child at this stage has therefore not yet acquired the notion of multidimensional quantity, owing to a lack of composition between the relationships of differences. For him the quantity of liquid does not depend on the combination of the various relations of height, cross-section, number of glasses, etc., since each of these relations is considered separately, as though independent of the others. Each relation therefore constitutes merely a 'gross quantity' that is essentially uni-dimensional." (Piaget, 1941, p.12)

Children begin to coordinate multiple variables as they progress to the second stage, marking the beginning of the proportionality schema. Piaget explains that although children become aware of the need to coordinate height and cross-section in volume judgements they cannot quantify these changes. What this means is that if the perceptual differences are slight then children can solve the problem; otherwise visual differences still form the basis of comparisons. Piaget called this process the acquisition of the logical multiplication of relations (p.22), stating that these children can only coordinate height and cross-section on the plane of qualitative seriation. So if one variable is controlled (e.g. cross-section) children can reason about differences in the second variable (e.g. height); however if height and cross-section vary in opposite directions (as in Piaget's conservation problem) children fail because they cannot yet quantify these differences.

Children at the second stage also make progress in understanding part/whole relations. Once children are able to create simple seriations between variables this leads to the idea of total quantity.

"These logical multiplications of relationships are essential to the child's solutions, since he cannot assign any numerical value to the two dimensions and is therefore unable to multiply them arithmetically. Moreover, this logical operation makes possible the conception of a further relationship, that of total quantity, which is the logical product of height and width." (Piaget, 1941, p.20)

As children move to the third stage, the schema of logical multiplication develops further, to what Piaget describes as the arithmetization of logical multiplication (p.22). At this point, initial problems with logical multiplication (question 1) and conservation (questions 2 and 3) are solved systematically. The shift from logical multiplication of relationships (in stage 2) to the arithmetization of logical multiplication (in stage 3) occurs when children develop the notion of unit. It is at this point that children quantify the changes and understand that when a liquid is poured into a glass that is narrower but taller than the original glass, the change in height is compensated for by the change in width.

Piaget's conservation experiment has received much criticism since its publication, for example, in relation to the age at which children can conserve (Gelman and Gallistel,

1978) or whether children conserve sooner if questions are posed by a third party (McGarrigle and Donaldson, 1974).

For the purpose of the current thesis these types of criticisms are not important. What is of importance are three points raised for the first time by Piaget and which have not been seriously disputed since. The first point is methodological: Piaget showed that to understand conceptual difficulty, logical requirements must be studied independently from perceptual cues. It was shown quite clearly that when young children consider relations between quantities there is a conflict between perception and logic. Very young children tend to reason on the basis of perceptual differences prior to the use of logic. This conservation experiment showed that the influence of perception on thinking continues even when children begin to understand logic. The need to distinguish perception from logic will be shown more clearly in later studies by Piaget, in which perception provokes correct responses which rarely lead to a lasting logical awareness. In short he highlighted the importance of creating a divergence between perception and logic when assessing conceptual difficulty, a concept vital to the design of the current thesis.

The second point raised by Piaget for the first time was that with relations between quantities problems such as volume, the initial challenge for children is to consider the relevant dimensions simultaneously. Piaget showed that children initially only pay attention to dimensions in isolation without coordinating them. What Piaget does not consider here is whether the notion of a relation between quantities problem alone makes the coordination between dimensions difficult, or whether the underlying structure of relations between quantities problems also plays an important role. This question is one which is pursued in the current thesis and will be returned to when discussing problem classification.

The third point raised is that success with relations between quantities problems begins with success on seriation problems. This point is also crucial to the current thesis and wider development of proportional reasoning. Piaget called this development the schema of logical multiplication of relations, and incorporated problems when one dimension is controlled and the second varies. This idea was only really touched upon in this initial experiment, as a point separating the notion of relations between quantity problems and their quantification. This initial work considered the logical multiplication of relations in a limited sense as the study was designed to look at conservation and not variation of relations between quantities. As an important component of the proportionality schema (which was the problem of wider interest), Piaget's interest in the logical multiplication of relations continued throughout his career. His later writing on proportional relations problems expanded on these initial observations to create the distinction between seriation problems involving direct or inverse relations. Although Piaget's assertion that part/whole problems are more difficult than seriation problems was another important observation, it is the question of seriations as the first significant development of the proportionality schema that is of direct importance to the current thesis. Piaget later studied the logical multiplication of relations in greater detail, this is considered in the next study presented; the projection of shadows experiment.

2.1.2 The Projection of Shadows Experiment (1958)

Inhelder and Piaget (1958) returned to the study of the logical multiplication of relations in a second series of experiments on the development of formal operations. Here the projection of shadows experiment provides a good example of how Piaget began to consider seriation problems in greater detail, looking particularly at the distinction between direct and inverse relations. Children observed a light projected onto a white screen with rings of varying diameters placed between the light and screen to cast a shadow on the screen. Piaget asked the children to find a rule explaining the distances at which two or more rings could project shadows of equal sizes. Children made predictions before the experiment and were asked to comment on the results. To pass the task children had to understand that the size of the projected shadow resulted from the relation between the size of the ring and the distance from the light source.

The projection of shadows experiment was similar to the conservation of continuous quantities experiment in that it was considering equivalences. The difference was that the first experiment was examining whether equivalence is maintained after an irrelevant transformation; whereas the projection of shadows experiment was investigating how children create equivalences with relations between quantities problems. Moving to this question allowed Piaget and Inhelder to consider in greater detail the logical multiplication of relations.

As with conservation, the projection of shadows experiment involves three dimensions that are related to the outcome (shadow size). The first is the distance between the light source and the screen; the second is the position of the ring; and the third is the size of the ring. As with the conservation of continuous quantities experiment, the problem was simplified by fixing the distance between the light source and the screen. The children had only to consider the role of two dimensions: the position and the size of the ring. Unlike in the conservation experiment, creating shadows of equivalent sizes requires an understanding that the relevant dimensions can be either directly related (i.e. if the distance from the light source is fixed then the larger the ring, the larger the projected shadow) or inversely related (i.e. if the ring diameter is fixed then the greater the distance from the light source, the smaller the projected shadow) to shadow size. It was Piaget and Inhelder's contention that conceptual understanding of shadow size as a relation between quantities problem could only occur when the logical multiplication of relations and, more specifically, direct and inverse relations were understood.

Piaget tested children aged 7–12 years, and observed their progress through developmental stages 2 and 3, culminating in the schema of proportionality. Piaget reports the reactions of children aged 7–11 years as at stage 2, with the youngest children viewed as being at sub-stage 2a. The first difficulty reported was similar to those found initially with conservation: that shadow size was a relation between quantities problem dependent on the relation between two independent dimensions (ring placement, ring size). It is important to recall that with conservation both dimensions (height and cross-section) were directly related to volume, and children's initial difficulties stemmed from considering only one dimension at a time. The dimension children focused on with conservation varied from child to child (or even between different questions posed to the same child). In this experiment though, the relation of each dimension to shadow size differs, with ring size directly proportional to shadow size, and ring placement inversely proportional. Piaget and Inhelder reported that children initially had a clear preference for considering the directly related dimension. At the start of the experiment children at stage 2a stated the direct relation between ring size and shadow size, but did not view the inversely related dimension (distance from the light source) as having any effect. Upon observing the experiment, stage 2a children saw that distance affected shadow size, and could then order the rings by size; these children relied on rough approximations of distance without establishing proportional relations.

Piaget reported that children aged 8–10 years then move into sub-stage 2b. From the start these children predicted both the direct relation of ring to shadow size and the inverse relation between distance from the light source and shadow size. Although children at stage 2b knew about the inverse relation, Inhelder and Piaget (1958) noted that at stage 2b children still preferred to think in terms of direct relations. Piaget and Inhelder reported that stage 2b children reformulated inverse relations into direct relations when adjusting erroneous predictions. Instead of considering the relation between shadow size and the distance between the ring and the light source, the children used the distance from the screen as a reference point: the greater the distance from the screen the larger the shadow. These observations suggest that the inverse relation represents a significant conceptual difficulty for children when asked to think about relations between quantities. Piaget and Inhelder also noted that although children at stage 2b have a greater qualitative understanding of the proportional relation between ring size and distance from the light source, the quantification of this relation does not show true proportions (i.e. multiplicative relations between quantities but the calculation of constant additive differences).

Piaget and Inhelder reported that the acquisition of proportional relations occurs at stage 3 (12 years) and is due to the child's use of propositional logic. Propositional logic is acquired through what Piaget refers to as the qualitative schema of proportionality (p.207). This means that the same product (e.g. shadow size) can result from two different causes (e.g. decreasing distance, increasing diameter or increasing distance, decreasing diameter). At stage 3 children understand that the relations are multiplicative rather than additive, because they can distinguish between and coordinate two types of transformation (inversion and reciprocity). Transformation by inversion refers to the idea that change to a variable can be cancelled; while transformation by reciprocity is the idea that change to a variable can be compensated without cancelling.

This study on the production of equivalent relations between quantities introduced the idea of direct and inverse relations. It was shown that children work more easily with direct relations. Direct relations were not only easier to discover (stage 2a), but preferred by children even when they understood the inverse relation (stage 2b). It was also shown that children first perceive the relation between variables as additive before they consider proportions as multiplicative. The projection of shadows experiment was about understanding geometry and the cone of light emitted from the light source. This is a difficult concept, one Piaget deemed too difficult to present to children younger than 7 years. It could be argued therefore that in this experiment the requirement of additional specialist knowledge made relational thinking more difficult than it might otherwise have been for the children. Like the conservation of continuous quantities problems the notion of relations between quantities problem was modified, with Piaget and Inhelder simplifying the problems to reduce the number of relations children had to think about. To consider how thinking about direct and inversely related dimensions develops in relations between quantities problems Piaget used a different research strategy. What follows is a discussion of Piaget's later work on speed demonstrating how Piaget moved from studying the production of equivalent outcomes with relations between quantities problems to studying how children think about differences.

2.1.3 The Direct Relation Between Distance and Speed (1970)

During his investigations into children's understanding of speed, Piaget (1970) considered direct and inverse relational thinking in separate experiments, which will be reported separately.

In the direct relations experiment the child observed two objects on parallel tracks with different starting points but identical finishing points. Both objects were set to travel for the same length of time. The child was then asked if one object had travelled with greater speed. If the child understood the need to coordinate distance and time in judgements of speed they should then be able to reason systematically that the object covering the greater distance during the time period was travelling with greater speed.

The speed comparison problems were specifically designed by Piaget (1970) to advance the argument first made during the conservation experiments that knowledge does not come from perception. The speed comparison experiments were designed to see what happens when important perceptual cues (overtaking and stopping points) conflict with the logic. He used problems with an initial convergence between the perceptual cues and logic, before creating increasing degrees of divergence; three scenarios in particular highlight how Piaget achieved this. In the first scenario the quicker object (on the longer track) visibly overtakes the slower object (on the shorter track). Here the perceptual cues are consistent with the logic as the quicker object overtakes and finishes ahead of the slower object. In the second scenario, the object on the longer track reaches the same point as the slower object by the end of the allocated time. In this second problem there is a divergence between the perception and logic; although the object on the longer path covers more distance the perceptual cues indicate the speeds were the same. In the third scenario the quicker object finishes just behind the slower object. Here the perceptual cues indicate a result opposite to the logic as the quicker object always remains behind the slower object. Piaget (1970) reports three stages of development leading to the systematic understanding of the direct relation between distance and speed by around 8 years.

During the first stage, only the first scenario (convergence between perception and logic) is judged correctly. Piaget explains that children's first intuitions about speed relate to an idea that movement denotes a change of position. The extent of children's reliance on perceptual cues was demonstrated by Piaget when looking at children's reactions to problem 2 (when both objects reach the same point). Piaget reports that children at stage 1 produce correct judgements if the difference in track lengths increases from a ratio of 2:1 to one of 4:1 or more. However, this newfound insight was also grounded in the perception of motion and not a sudden logical development:

"...let us note that when the disparity in the distances travelled becomes too great, the child acquiesces and recognises which of the cars goes faster. But in that case it is either simply a question of perceptual observation of speed without proving this statement by the greater distance travelled or else it is a kind of unspecific intuition such that one maximum invokes another without particular reference to the road followed as a distance." (Piaget, 1970, p.162)

The key finding at this stage was that children do not try to coordinate distance and time in judgements of speed. To stage 1 children, speed is all about perceptual differences and acts of overtaking.

During stage 2, children were more successful at judging speed although this success remained rooted in a perceptual concept of speed, with the need to observe overtaking remaining the central argument. Children at stage 2 have acquired the ability to use the idea of overtaking more abstractly to predict the future placement of objects. This leads to success with scenario 2, where the quicker object catches up to the slower object. Children at stage 2 solve this by thinking that overtaking would occur if the objects had continued moving.

Although children at this stage are basing their judgements on perception, they are beginning to pay more attention to the unequal starting points and the idea that one object must travel further to catch up to the other. As the notion of overtaking is still central to children's thinking at this stage, the third problem (when the quicker object fails to catch up) remains too difficult as children fail to comprehend that a quicker object could finish in second place.

Children progress to stage 3 at around 7–8 years and are able to solve direct relations problems systematically. Now children abandon overtaking as the only factor in judging speed in favour of considering the appropriate relation between distance and time.

2.1.4 The Inverse Relation Between Time and Speed (1970)

Piaget proceeded to test children's understanding of the inverse relation between time and speed when distance is controlled. The experiment was similar to that used for direct relations problems: children observed two objects travelling along tracks of equal length with varying start times. Of the various scenarios presented to children, two mirrored quite closely the divergence problems used in the direct relations experiment. In the first scenario, both objects leave their starting positions at different times but reach the end at the same time. In the second scenario the quicker object does not quite catch up to the slower object.

Piaget (1970) noted three stages of development corresponding closely to the stages observed for direct relations problems. During stage 1, as with the direct relations scenarios, overtaking must occur for children to report a difference in speed. Children argue that when finishing points are equal, speeds must also be equal and when finishing points are different, finishing further ahead indicates a quicker object. As with direct relations problems, varying one dimension significantly led children temporarily to the correct judgement. Stage 2 is perhaps more interesting as children progressively reach the correct answer through trying to incorporate time differences with their preferences for overtaking. At first, children attribute greater speed to the object starting first (because it was able to get ahead). They then attribute equal speed to both objects, before correctly inferring that as the times taken for the journeys differ so must their speeds. As with the stage 2 children on direct relations problems, these insights are based on a perceptual, not logical, argument.

"How does the subject achieve this correlation when he is not helped to break down the factors? It is through the equating of making up on lost time and catching up on space still to be covered. The latter schema being comparable in turn with overtaking." (Piaget, 1970, p.175)

During stage 3, again at around 7–8 years, children are able to solve this inverse speed problem immediately because they are able to coordinate distance and time into speed judgements.

The speed experiments highlight three points. First, young children hold strong convictions about speed. Far from considering one dimension randomly as was the case with conservation, children reason on the basis of a general idea about overtaking, although this was only relevant to a few situations. One of the unique features about intensive quantities is that many examples occur in children's day-to-day lives. The perception of speed differences is something most children experience without explicitly considering distance or time. It is perhaps not surprising that children try to use this knowledge when faced with more formal speed situations.

The second point argued convincingly here was that knowledge does not come from perception. Piaget was able to show that increasing perceptual differences can make children pay attention to a different dimension, for example getting children to think about distance when the ratio of track length was increased from 2:1 to 4:1. He also showed the way children use perceptual knowledge to solve divergence problems. These perceptual solutions never led to a consistent and systematic understanding of the logic.

Thirdly, although Piaget (1970) did not make direct comparisons between direct and inverse relations, reports from these independent experiments place the acquisition of both ideas at around 8 years. The coordination of previously independent dimensions is the key difficulty. Whether these dimensions are directly or inversely related mattered very little to Piaget. This point was further developed in Piaget and Inhelder's (1975) quantification of chance studies, which considered direct and inverse relations problems as equivalent single variable problems. However, this collection of studies is at odds with Piaget's own observations on the projection of shadows experiment. In short, the problem of direct and inverse relations is one Piaget did not solve, and which requires more detailed study.

2.1.5 General Conclusions from Piaget's Studies

Reviewing Piaget's work has shown that while he addressed particular questions systematically, others received less attention, and some potentially significant distinctions were not considered at all. A major issue touched upon by Piaget but not addressed as systematically as other issues was that of whether the distinction between direct and inverse relations represents a significant conceptual challenge. When Piaget examined children's solutions to seriation problems, he showed, notably in the projection of shadows experiment, that directly related dimensions are easier for children to deal with than inversely related dimensions. Due to results in some of his other experiments, Piaget's stance in later speed experiments changed to include an implicit assumption that children solve direct and inverse seriation problems at roughly the same age. In his later chance experiments Piaget explicitly regarded direct and inverse relations as one type of problem (single variable problems). Piaget's main aim throughout these experiments was to explain the wider schema of proportionality. As such Piaget saw direct and inverse relations as part of the same logical schema (logical multiplication of relations). This is why he did not pursue the systematic comparison of these problems. Piaget's experiments therefore leave the issue of direct and inverse relations unresolved. This issue is pursued as one of the aims of the current thesis and is considered in greater detail when reviewing post-Piagetian work in Section 2.

A second area of ambiguity arising from Piaget's focus on describing the logical schema of proportionality was whether the nature of the relations between quantities problems he was testing represents distinct conceptual challenges. Piaget presented many examples of relations between quantities problems but he never questioned whether distinctions such as that between extensive quantities (e.g. volume as used in the conservation experiment) and intensive quantities (e.g. speed, chance) were conceptually important. This issue is all the more perplexing as Piaget was the first developmental psychologist to introduce an explicit definition of extensive and intensive quantities, in his first major work on children's understanding of number (Piaget, 1941). The question of whether extensive and intensive quantities represent distinct conceptual challenges forms the main aim of the thesis. This issue is also discussed in greater detail during the section on problem classification presented in Section 2.

2.2 Section 2: Post-Piagetian Research on the Development of Children's Relational Thinking with Relations between Quantities

There are two aims in this section. The first is to examine how the distinctions made during Piaget's pioneering experiments on the origins of relational thinking were considered by the work which followed his. The second aim is to consider the role of problem classification as a basis of conceptual difficulty, an area Piaget did not look at during his research.

The section is therefore presented in two parts. The first part covers post-Piagetian research on the coordination and seriation of relations between quantities, identified by Piaget as children's initial hurdles. Here the focus is on whether the distinction between direct and inverse relations is conceptually important. The studies reported in the first part are discussed in two sets, divided by whether the problems relate to intensive or solely extensive quantities. This will help situate the discussion in part two, which looks beyond the work of Piaget to the question of problem classification.

2.2.1 Relational Thinking in Intensive Quantities Contexts

In relation to the aims of the thesis there are two assumptions which can be drawn from Piaget's writing. The first is that to fully understand the importance of referent quantities in relations between quantities problems, children must know that the referent quantities are both directly and inversely related to the third extensive or intensive quantity. The second is that these ideas are expressed systematically at around 8 years. As with many of Piaget's experiments, it is not necessarily the descriptions of children's logic but the timing at which children acquire certain abilities and the extent to which children's reasoning is influenced by task variables that have been disputed.

2.2.1.1 Post-Piagetian Research on the Timing of Intensive Quantity Understanding

Studies in the proportional reasoning literature involving intensive quantities claim that children's ability to coordinate relations is evident earlier than Piaget had originally suggested. For example, Singer, Kohn and Resnick (1997) claimed that young children can consider multiple dimensions simultaneously under certain conditions.

Singer et al. (1997) demonstrated this by presenting children with density problems using crowdedness as the target setting (see Figure 2.1). The problems took two forms; in the first both dimensions (area of flowerbed and number of flowers) were presented in combination, while the second form presented the dimensions separately. With combined dimensions children as young as 5 years were successful. With separated dimensions the task became more difficult, with many of the younger children reasoning solely on the basis of the number of flowers.

Figure 2.1 Density problems with combined and separated dimensions

Singer et al. (1997) claimed that this was evidence that young children could think about proportional relations. However, it is clear that these researchers' demonstrations of early ability are no different from the type of convergence between logic and perception problems that Piaget created in his speed experiments. It must also be noted that the items used by Singer et al. present positive perceptual cues from both dimensions. It can only be said, therefore, that under the most favourable perceptual conditions (when both dimensions pointed to the same result) 5 year olds made accurate density judgements. Also, the problem presented with separated dimensions was very difficult to solve through reasoning alone.

With this problem both the area and the number of dots varied, making it a difficult computational problem. A fairer comparison problem would have been to hold one dimension constant (dots or square size) while varying the second. This could have revealed more about the discrepancy between solving perceptually based and logically based density problems. Although here it is clear that early success was based on perception and not logic, there are other studies which have fallen into this trap more subtly.

An earlier study on speed problems by Wilkening (1981) attempted to analyse separately children's relational thinking from their perception of motion. He asked 135 participants taken from three age groups (5–6 years, 9–10 years and 17+ years) to solve problems requiring the integration of distance, time and speed. To achieve this, the participants were presented with scenarios in which information of two quantities was known with a judgement required on the third. There were two tasks of direct relevance to the current thesis. The first looked at judgements of the direct relations between distance and speed when time was controlled. The second looked at judgements of inverse relations problem between time and speed when distance was controlled. For each task, the participants were shown a similar scenario in which a dog lived on one side of a long footbridge (used to represent a measure of distance). Three animals of varying speeds were used (turtle, guinea pig and cat) and a tape with the sound of barking provided the measure of time. In the direct relations task the tape of the dog barking was played for a set length of time after which children judged the distance different animals had travelled along the bridge. For the inverse relations task, animals of various speeds were placed at set finishing points and Wilkening asked the children to play the barking tape as long as they felt it would take each animal to reach the predetermined point – the quicker the animal, the shorter the time the child would need to play the tape.

Wilkening (1981) found that all groups made accurate judgements about direct relations between distance and speed. This suggests that even by 5 years children had a good awareness of the direct relations about speed.

On inverse relations problems differences were observed between the reasoning of the three age groups. In terms of a general understanding of the inverse relation, all three age groups correctly judged that slower animals would need more time to reach a fixed distance. However, the younger children failed to understand that the amount of time needed would increase proportionally as the target distance increased. Instead, 5 and 6 year olds opted to keep a fixed time difference between animals of different speeds at all three distance points judged (70, 140, 210cm). From this Wilkening concluded that some aspects of inverse relational thinking were not beyond young children.

However, the extent to which the children in Wilkening's experiment had a logical awareness of the inverse relation between speed and time is questionable. This is due primarily to the problem of presenting situations that contained perceptual cues. For example on inverse relations problems the children were shown animals individually and asked to play the tape for as long as it would take the animal to reach the fixed point. Wilkening (1981) explains that children used eye movement to keep track of the animal's progress. For a quick animal, children moved their eyes quickly, while for a slow animal the children moved their eyes slowly. Therefore, this primitive strategy may have produced the correct answer but children were solving the problem with a directly proportional perceptual variable – the quicker the animal, the quicker the eyes need to move. It is difficult to conclude from this study that children really understood the inverse relation.

A further study looking specifically at the development of children's direct and inverse relational thinking about speed was conducted by Acredolo, Adams and Schmid (1984), who presented similar scenarios as Wilkening to 6 to 11 year olds. Unlike the children in Wilkening's (1981) study children did not have to produce an answer by playing a tape or stating an exact distance, but instead had to reason explicitly about what varying different quantities meant in the relation between distance, time and speed.

Of interest to the current discussion were two items: the first in which two animals ran for the same length of time but covered different distances (direct relation); the second in which the animals ran for different lengths of time but covered similar distances (inverse relation). Significant differences were observed in children's general performance, attributed by the authors to a difficulty in understanding the inverse relation.

This study contrasts with the earlier conclusions from Piaget's speed experiments where direct and inverse relational thinking were observed at roughly the same stage. It is important to note though that Acredolo et al.'s (1984) direct item may have been solved by younger children because they did not control for perceptual biases. In particular they failed to observe the perceptual influence of varying finishing points originally reported by Piaget (1970). Further examples of problems which children solved perceptually rather than logically are reported in Wilkening and Huber's (2004) chapter on intuitive physics.

Even when perceptual cues are removed there are still some instances reported that suggest that quite young children have some capacity to reason about inverse relations, for example in concentration problems. Noelting (1980a, 1980b) studied how children reason about the taste of orange squash when different numbers of glasses of orange squash and water are mixed. He interviewed 321 children and adolescents, aged 6–16 years, in one-to-one (for younger children) and whole class (for older children) settings. The children were given 23 items involving the comparison of taste through interpretation of orange squash/water ratios. Items were displayed using small glasses of orange squash and water (during one-to-one testing) or with pictures in booklets showing the orange squash/water ratios as glasses of each (in group testing). For each item, the children had to decide which of the two orange squash/water mixtures would have a stronger taste or whether both drinks would taste the same. Of interest to the current thesis are Noelting's non-computational items. Unfortunately, Noelting (1980a) did not include in his analysis direct relations problems where one of the variables was controlled, although three inverse relations problems were posed. These comprised the comparison of the following orange squash/water ratios: 1/0 vs. 1/1, 1/2 vs. 1/5 and 1/1 vs. 1/2. Noelting reported a high percentage of 6 year olds (78%) and 7 year olds (65%) passing all three items, with roughly 92% of children at these ages able to pass at least one of these items. Noelting (1980b) reported that children who were unable to pass these items usually failed because they compared drinks only on the amount of orange squash used; this type of error following Piaget's earlier work on chance was termed centration:

"(before considering amounts of water) The strategy applied to solve the item is the sole comparison of the first terms (centration on one relation)." (Noelting, 1980b, p.345)

Noelting's (1980a 1980b) work suggests that inverse relations in some intensive quantity settings are not beyond the scope of many 6 year olds, though this conclusion must be solely tentative as the children tested may have been cued to consider variations in water during a series of orientation problems. One of the orientation problems the children received was to compare orange squash/water ratios (1/1 vs. 1/4). If children gave the wrong answer the following discussion was always initiated:

"And the water, do you think it is important? What does it do to the taste? What happens if I add more water (add still two or three more glasses of water on side B)?" (Noelting, 1980a, p.222)

By dedicating such attention to water for children who failed this item it is possible that children decided to consider the amount of water in the problem not because they understood the inverse relations but because the questioning during orientation had led them to believe that it must be important.

Controlling for language is a further issue making it difficult to establish the extent of children's skill with relational thinking. This problem is highlighted in the work of Strauss and Stavy (1982), who looked at children's relational thinking about taste. They assessed the competence of 180 children aged 4–12 years on both direct and inverse relations problems regarding the taste of sugar/water solutions. They found children as young as 5 years (95%) were able to understand the direct relation that existed between sugar, water and the sweetness of taste. Strauss and Stavy also observed a clear delay in competence with the inverse relation as levels of success remained low until age 9, when 70% of children correctly inferred that controlling the amount of sugar then adding more water makes a drink tastes less sweet.

Although Strauss and Stavy's (1982) study highlights the difficulty of inverse relations problems it provides another example of how methodological problems may contribute to reported differences. As Strauss and Stavy were interested in children's ideas about sweetness rather than a general understanding of taste they asked children to think about direct relations only when the quantity of water was controlled and the quantity of sugar was varied. This meant that on the inverse relations item they controlled the amount of sugar and varied the amount of water, which creates a confound between relation and quantity varied. It is unclear whether children had a genuine problem understanding inverse relations or whether children's difficulties were attributable to a problem of variable salience with sugar.

Desli (1999) (reported in Nunes, Desli and Bell 2003) also made comparisons between direct and inverse relations within the contexts of taste and cost while controlling for the problem of language experienced by Strauss and Stavy (1982). She asked 105 children aged 6–8 years to work on a series of non-computational tasks requiring direct or inverse reasoning to solve. The first context mirrored closely the problems used by Strauss and Stavy with children being asked to think about what happens to the taste of lemon juice when sugar is added. The second context was the relative cost of various purchases when either quantity of product or money paid was varied. In both contexts Desli controlled the language used in the problems. For example, when children were shown a problem in which the amount of lemon juice was controlled and the amount of sugar varied between two drinks, Desli asked children both about the relative sweetness (direct) and sourness (inverse) of the taste. The results showed that after controlling for language on both taste and cost problems, performance on direct relations items was at ceiling level. When inverse reasoning was required the proportion of scores above chance fell to 10%, 26% and 60% for 6, 7, and 8 year olds respectively. Unlike the studies presented previously, Desli provides the first indications that when no perceptual information is available inverse relations with intensive quantities are difficult for children.

2.2.1.2 The Influence of Task Variables on Children's Thinking about Intensive Quantities

One of the many directions taken by post Piagetian researchers was to apply quantitative designs to Piaget's qualitative descriptions. The result of this attempt to quantify Piaget's developmental stages was a body of research asking whether varying the presentation of Piagetian tasks influenced children's performance. From this area of research there are two forms of task variation of interest to the current thesis on children's relational thinking. The first task variation is whether the presentation of quantities using either qualitative or quantitative descriptions significantly affects children's performance. The presentation of quantities is of particular interest because direct and inverse relational thinking problems can be solved without the need for computations. For example, when thinking about the taste of orange squash the extensive quantities of water and orange concentrate can be described without specific numerical measurements using relational terms such as more, less, same. The presence of qualitative and quantitative information on children's thinking is a question raised in the current thesis, as it remains largely untested in the domain of direct and inverse relational thinking about intensive quantities. This is in contrast to the numerous studies conducted within the context of mixing and sampling of intensive quantity problems which have highlighted qualitative vs. quantitative presentations as an important task variable. The second type of task variation from the general proportional reasoning literature of interest to the current thesis regards the problem structure itself and the need to consider direct and inverse relations problems using both comparison and missing value proportions problems.

2.2.1.3 The Impact of Qualitative vs. Quantitative Descriptions on Children's Success with Intensive Quantity Problems Evidence from Sampling and Mixing of Intensive Quantity Studies

The primary reason for the interest shown in qualitative vs. quantitative forms of presentation in the sampling and mixing literature is because sampling and many mixing of intensive quantities problems are solvable without the need to use computations. Although the presence of problems with qualitative or quantitative information does not change the underlying logic of these problems, research in this area has consistently shown that children's thinking about mixing and sampling intensive quantities is significantly affected by this task variation.

The work of Cowan and Sutcliffe (1991) is representative of a typical study from the mixing of intensive quantities literature. They looked at how a sample of 83 children (7-12 year olds) worked with temperature mixing problems when asked to consider what happened when two glasses of water at various temperatures were mixed. Cowan and Sutcliffe's independent groups design assigned children to either a qualitative or quantitative descriptive condition. For example children in the quantitative group were asked what would happen if a glass of water at 30°C was mixed with a glass of water at 30°C. The children in the alternative qualitative condition were provided with the descriptors (hot, warm, cool, and cold) instead of numbers. Cowan and Sutcliffe results showed that when qualitative descriptions were used children made fewer errors. The reason for lower levels of task performance in the quantitative condition was because a greater proportion of the children made additive errors (treating the problem of 30°C mixed with 30°C as 30 + 30). Later studies have replicated these findings (i.e. Dixon and Moore, 1996; Desli, 1999).

Ahl, Moore, and Dixon (1992) posed a similar question when considering the influence of intuitive vs. numerical knowledge. Ahl, Moore, and Dixon's study differed from Cowan and Sutcliffe's (1991) design as the form of presentation (qualitative or quantitative) was this time treated as a repeated measure. In this study half of the children received the qualitative version of the task before the quantitative version and vice versa. The prediction of the authors was that the children receiving the qualitative problems would get a more intuitive feel for the task making them less likely to use an incorrect additive strategy on problems with quantitative descriptions. They worked with 224 participants from 3 age groups 10-12 years, 13-15 years, 18-22 years. Ahl, Moore and Dixon's (1992) results confirmed their original predictions:

"Performing the intuitive version first decreased the likelihood that numerical temperature would be treated as an extensive rather than an intensive quantity." (p.81)

When looking at the reasoning of their participants with the younger age groups (comparable to the older children from Cowan and Sutcliffe's, 1991 study) Ahl, Dixon, and Moore (1992) found that children given the numerical condition first were more likely to make errors based on treating the intensive quantities as extensive quantities by relying on additive strategies.

One problem of using temperature as a general indicator of how children think generally about intensive quantities comes for the unique way in which temperature is synonymous with the temperature scale. Temperature expressed in terms of the Celsius scale for example could confuse children into thinking temperature behaves as an extensive quantity. This is because the Celsius scale gives no indication that temperature is composed of the ratio between (kinetic energy and mass). There are examples of other intensive quantities which are known by their scale, but these tend to keep explicit their referents, i.e. speed is generally referred to as miles per hour. In short the negative impact of quantitative presentations of temperature may say more about the global use of the temperature scale which resembles an extensive quantity scale.

The sampling of intensive quantities literature has generally avoided the problem of using intensive quantities which appear similar to extensive quantities in the way they are referred to by working with intensive quantities for which the extensive quantities are explicit. The most common form of sampling problem in the literature to meet these criteria is work on concentrations within the setting of taste. For example, Harel, Behr, Lesh and Post (1994) looked at the ability of 11-12 year olds to solve intensive quantity sampling problems. The children were presented with a picture of a carton of orange juice and two glasses of juice, the carton of orange juice had a label stating there were 40oz of water and 24oz of orange concentrate in the carton. The two glasses were labelled 7 and 4oz respectively. The children were told that the orange juice had been poured from the carton into the glasses and were asked whether the 7 and 4oz glasses would taste the same. Harel et al found that the quantitative information in the problem led children to try and solve the problem with computations and numerical comparisons which led them to incorrect 'different' taste responses. The pattern of responses was more varied than that shown in the temperature mixing studies. Some children in Harel et al's study believed the 7oz glass to taste more orangey as it would contain more orange, while other children stated the 4oz glass would be more orangey as it would contain less water.

The negative impact of numerical information on sampling problems was also shown by Schwartz and Moore (1998). Schwartz and Moore made a direct comparison of qualitative vs. quantitative forms of problem presentation. They report that children (11-12 years) receiving taste sampling problems (in the form of diagrams) accompanied with quantitative information made a higher proportion of incorrect additive comparisons, than children receiving the same problems accompanied with qualitative descriptions (i.e. big glass, small glass).

The literature on mixing and sampling intensive quantities provide clear evidence that in these settings children's reasoning is negatively affected by the presence of quantitative descriptions of the quantities. Even when the problems were presented with clearly defined extensive quantities as with the taste mixing problems children receiving quantitative descriptions used the numerical information as a cue to attempt computations. The direct and inverse relations problems under investigation in the thesis share a key similarity with the problems posed in the mixing and sampling literature; no computations are required to reach the solution. The impact of qualitative vs. quantitative forms of presentation on direct and inverse comparison problems though is so far untested. The current thesis will make this comparison systematically by asking firstly whether this distinction matters in terms of children's general performance on directional reasoning tasks, and secondly whether the use of qualitative and quantitative descriptions present different challenges to children's relational thinking if the problems presented involve intensive quantities or are composed solely of extensive quantities.

2.2.1.4 The Impact of Missing Value Problems on Children's Proportional Reasoning Success

The second task variable from the post Piagetian period considered in the current thesis is the use of missing value proportions problems. There are two reasons why this is deemed a potentially relevant task variable for the problems investigated in the current thesis. The first reason is that although the studies of the thesis have been designed within the paradigm of the comparison methodology, researchers such as Fujimura (2001) report that when studies are compared on the basis of using either comparison or missing value problems to assess proportional reasoning skill then:

"*....comparison problems which require the solver to compare two ratios appeared to be corrected solved at a later age than missing value problems, which require the solver to apply a proportional relationship to calculate a number.*" (p.589).

The conclusion drawn by Fujimura (2001) is based on looking at children's performance between independent studies and when complex versions of comparison problems requiring computations are presented to children older than those participating in the current thesis (e.g. Noelting, 1980). The current thesis makes a direct comparison between comparison and missing value problems, with younger children on simplified problems.

The second reason for including missing value problems as an important task variable stems from work suggesting missing value problems reveal patterns of development not possible from the comparison method alone. The main proponents of this view are Wilkening and Anderson (1982) who suggested that relying solely on the comparison problems to make inferences about children's development can lead to results which fail to capture important aspects of development. Wilkening and Anderson worked with the Piagetian balance scale task (Inhelder and Piaget, 1958) to compare the inferences possible about the development of children's competence when using either the comparison task paradigm of Siegler's rule assessment technique (Siegler, 1976) or their own functional measurement paradigm which required children to use computations.

The work of Siegler (1976) was chosen as it was typical of the direction taken by early post Piagetian researchers to apply qualitative methodology to assess the key concepts of Piagetian theory. The balance scale task was one of the more famous Piagetian tasks in which children were shown a balance scale for which two variables could be manipulated; the weight placed on each side and the distance of the weight from the fulcrum. The child was then asked to compare both sides of the scale and to predict whether the scale will balance and if not which side would go down. By presenting children with a series of carefully constructed comparison problems Siegler developed a theory which reduced Inhelder and Piaget's descriptions of balance task performance to four progressive rules. The first rule was to make judgements only on the basis of weight (ignoring the role of distance). The second rule was to make judgements based on the distance of the weight from the fulcrum, only when the weights on each side were equal. The third rule included the additional step of 'muddling through' if distance and weight information conflicted (i.e. if one side had the greater weight, while the other the greater distance). The fourth rule involved children correctly applying the concept of torque to the variables in the balance scale.

Although Siegler (1976, 1978) presents evidence to support the existence of these rules, Wilkening and Anderson (1982) pointed out that Siegler's observations of children's task performance often conflicted with the subsequent explanations the same children provided. The primary discrepancy between children's choices and justifications was that children's choices on the task often led to the conclusion that the child only paid attention to one variable at a time (rule two), while their verbal justifications suggested they accounted for both variables simultaneously. In short the verbal responses of many of Siegler's rule two users were consistent with those of a typical rule three user.

Wilkening and Anderson (1982) proceeded to test children's (6-12 years old) ability to produce computations on balance scale problems. They achieved this by manipulating the weight and distance on one arm of the balance, they then asked the child to adjust the distance of the weight from the fulcrum on the opposite arm to the point they expected both sides would balance. The results of this experiment confirmed Wilkening and Anderson's appraisal of Siegler's (1976) rule assessment methodology. On the strength of choice task data children received rule two classifications while on the missing value computation task the same children showed more advanced rule three users and in some cases rule four users by incorporating both the weight and the distance from the fulcrum in their answers.

Wilkening and Anderson's study led them to make two general conclusions on the contribution of their missing value problems to the understanding of children's development on the balance scale task. The first was that the missing value computational problems showed children were able to consider multiple variables sooner than had been possible to show when relying solely on data from comparison problems. The second conclusion was that presenting children with missing value problems revealed aspects of development not visible with Siegler's comparison task. Wilkening and Anderson's method allowed them to shed more light on the 'muddle through' descriptions in Siegler's (1976) theory. Performance on missing value problems showed this muddling through consisted of a primitive additive combination of variables, while more advanced children muddled through with a mixture of additive and multiplicative strategies.

Wilkening later posed the question of whether increasing the computational difficulty of a problem facilitates children's ability to attend to multiple variables (Jäger and Wilkening, 2001). Jäger and Wilkening report a replication of Desli's (1999) paint mixing task in which they asked children (8-10 years) to predict the final hue of a red paint when different (light and dark) shades of red paint were mixed.

Jäger and Wilkening (2001) present data from two experiments the first followed closely Desli's (1999) original design in which equal quantities of dark and light paint were mixed. Children were then asked to choose from an ordered set of possible hues, the final colour of the mixed paints. As this task was about finding the medium hue children passed these items by selecting the colour half way between the dark and light shades that the experimenter presented. In this experiment the children produced what was referred to as an adding rule that is choosing a colour darker than the dark paint mixed. For example if a light red (two) was mixed with a darker red (eight) the child would choose an even darker red on the scale (ten) as the final mix colour (light and dark makes a darker shade of the colour).

In the second experiment Jäger and Wilkening designed problems which they expected to be more difficult, but which were also expected to stimulate children to think about the role of volume when mixing paints. The problems were designed so that the children had to think about what happened when different measures of light and dark paint were mixed (i.e. a full tin of a light red paint mixed with half a tin of a dark red paint). In this experiment when children had to think about the volume of paint as well as the shade to arrive at the weighted average the additive rule prevalent in experiment one disappeared, to be replaced by a primitive form of the correct averaging rule. Jäger and Wilkening (2001) referred to as the 'range principle' (p.340). Children using the range principle understood that the final colour of the mixed paint must lie within the range of the two shades which were mixed. This principle was consistently violated by the additive rule users working on the simplified problems presented in experiment one. The conclusion reached by the authors was that increasing the complexity of the variables in the problem helped children to involve volume in their solution strategies and to think about what variations in volume mean rather than just adding as was the case in the simpler problems presented in experiment one. Jäger and Wilkening contend that increasing the complexity of the task prompted children to be more considered in their solutions and less likely to produce shortcut responses.

For the purpose of the current thesis the inclusion of more complex missing value problems into the design will provide an important accompaniment to the comparison methodology which forms the main methodological approach taken in the thesis. The inclusion of the missing value problem format has the potential to detail further how children work with the variables in relational thinking problems. The inclusion of missing value problems which are directly comparable to the comparison problems will also provide the first systematic comparison of these methods on younger children's ability to solve basic direct and inverse relations problems. It will also be possible to test systematically whether this methodological variation produces different results when problems are presented to children involving either intensive quantities or when problems only involve extensive quantities.

2.2.2 Children's Understanding of the Relations Between Extensive Quantities

One possible explanation for children's difficulties with relational thinking about intensive quantities could be the number of dimensions children have to consider. Therefore it is important to consider whether it would make a different if all the dimensions under consideration were extensive quantities. Although no studies have yet tested this distinction explicitly, there have been studies carried out which involved proportional relations between extensive quantities. These studies tend to come from two areas. The first area is the attribution theory literature and stems from studies looking at judgements of ability, effort and outcome. The second area is represented in the literature on sharing and division. Important evidence from both areas will be used to consider children's relational thinking when proportional relations involve three extensive quantities.

2.2.2.1 Direct and Inverse Relations from the Attribution Theory Literature

Starting with studies from the attribution theory literature, Karabenick and Heller (1976) looked specifically at children's understanding of the inverse relation that exists between ability and effort when outcome is controlled. They tested 128 children and adults from four age groups (6–7 years, 8–9 years, 10–11years and 18+ years). The participants were presented with a series of items; in each one they compared the effort or ability of two children who had just completed a puzzle. For example, information was given about the relative ability of the children (very good vs. very bad at puzzles) from which a judgement had to be made on effort (which child would have to work harder to solve the puzzle). The problems were presented to the children orally and to the adults as a paper and pencil task. Karabenick and Heller found that each age group solved inverse relations items significantly above chance. The children found reasoning about the inverse relation between effort and ability (more effort means less ability) more difficult than reasoning about the inverse relation between ability and effort (more ability means less effort). This led to the conclusion that:

"...by first grade (6–7yrs), children demonstrate some appreciation of the inverse relation between effort and ability. Further, it appears that inferring effort expenditure from ability and outcome information occurs developmentally prior to making ability attributions from effort and outcome information." (Karabenick and Heller, 1976, p.560)

Although this shows that young children may have some notion of inverse relations, Karabenick and Heller did not provide comparative data about children's direct relational thinking. There was also a lack of error analysis, making it difficult to understand why judging effort from ability was more difficult for children.

A later study was carried out by Kun (1977) with children aged 5–12 years that considered direct and inverse relations in children's understanding of causal schema in relation to achievement. Kun (1977) observed that the direct relation could be inferred by children as young as 5 years, while inverse relational problems proved difficult for children until around 9 years. Consistent with Karabenick and Heller's findings, Kun found that the inverse relation was more difficult for children to reason when more effort implied less ability (more effort and same outcome mean less ability) than when effort had to be judged from ability (more ability and same outcome mean less effort needed). The errors made by children on inverse relations problems were reportedly the result of systematically applying a 'halo schema', that is, children applied direct relations to inverse relations problems. This finding was later replicated by Surber (1980).

The important lesson from these studies on attribution theory is the report of an asymmetry in children's reasoning about different inverse relations. Though children were willing to infer that less able people need to put in more effort to solve problems, they were less likely in these settings to say that higher effort is only synonymous with low ability. This highlights the importance of designing problems that specifically test children's knowledge of how different quantities are related in relations between quantity problems in order to fully assess how children coordinate dimensions.

The attribution theory studies also show the difficulty of trying to make inferences between culturally and scientifically based concepts. Intensive quantities such as those described in the previous section are scientifically based concepts which consist of dimensions with fixed relations to one another. Effort, ability and achievement are culturally dependent concepts which behave quite differently. Though it is possible to generate inverse relations scenarios with effort, ability and achievement the validity of these relations is less clear. This is because with attribution theory one is straying from the path of logico-mathematical thinking into the area of cultural beliefs. More recent attempts to study attribution theory in children fall foul of the same difficulty (e.g. Thompson and Siegler's studies of children's understanding of basic economic principles, 1998 and 2000).

Therefore, to try to understand more clearly how children think about direct and inverse relations in extensive quantity settings, less complex settings must be considered. The review of the literature moves on to research conducted in the context of children's ideas about division and sharing as these studies provide examples of less complex extensive quantity relational thinking problems.

2.2.2.2 Direct and Inverse Relations with Sharing Problems

Sharing problems generally ask children to consider relations between three extensive quantities. It is necessary for the children to consider the total quantity to be shared, the number of recipients, and the size of each recipient's share. On the surface the size of each recipient's share seems to be an intensive quantity as it expresses the relation between the number of recipients and the total quantity to be shared. It is in fact an extensive quantity as it adheres to the part/whole relations of an extensive quantity. With sharing problems, the total quantity to be shared is equal to the sum of the individual shares, in contrast to the part/whole relations of an intensive quantity such as taste, whereby the parts and the whole remain the same (pouring a drink does not change the taste).

Sophian, Garyantes and Chang (1997) looked at children's understanding of sharing to see whether they could find the inverse relation between the number of parts to be shared and the number of recipients. Sophian, Garyantes and Chang presented 56 children (5–7 years) with problems depicting two sharing situations, varying either the number of objects (pizzas) to be shared or the number of recipients (monsters). The children had to choose which of two alternatives would result in more pizza for each monster. When the number of pizzas varied children had to think about the direct relation (more pizza means a greater share for each recipient). When the number of recipients varied children had to think about the inverse relation (more recipients means less pizza each). Sophian Garyantes and Chang reported that children solved significantly more direct than inverse relations problems. When looking for an explanation of the difficulties with inverse relations problems Sophian, Garyantes and Chang (1997) considered the consistency of children's reasoning. They found that only 27% of the participants consistently solved inverse relations problems in terms of direct relations above chance. This rate of consistency is low considering a forced-choice task was used with children given only two choices. The authors concluded that children within the range tested (5–7 years) did not understand inverse relations with fractional sharing problems and did not have a consistent strategy when faced with inverse problems. Sophian, Garyantes and Chang (1997) proceeded to carry out a second experiment testing the idea that the fractional context of the problem was behind children's difficulty with inverse relations and sharing. The authors tested whether presenting problems in a simplified subtractive context would result in an increased awareness of inverse relations. Working this time with 5 year olds (N=16) they tested children's understanding on inverse relation between two dimensions only (the more you give away, the less you have left for yourself). They presented this problem within the setting of a pizza monster having to fill up plates of food for his babies before he could eat the remaining food. The idea was that the more babies the pizza monster had, the more food he would have to give out and the less food he would have for himself. Within this two-dimension context children performed significantly better on inverse relations problems than the 5 year old children working within the relations between quantities context. One problem with this experiment though was the use of a forced-choice design in which children were not able to give the response that the same amount of pizza results in equal-sized shares. This meant that a possible conceptual difficulty which might arise under normal conditions was excluded.

A further study by Correa, Nunes and Bryant (1998; first reported in Nunes and Bryant, 1996) also looked at inverse relations in a sharing setting (sweets shared among different groups of rabbits). Children here found inverse relations significantly more difficult than direct relations. Although the children made very few mistakes when solving direct relations problems, the percentage of children solving inverse relations problems significantly above chance was relatively low for 5 and 6 year olds (30% and 55% respectively), this figure rose to 85% for 7 year olds. Correa, Nunes and Bryant (1998) looked in detail at the types of errors the children made answering questions about inverse relations between age groups. Unlike the items presented by Sophian, Garyantes and Chang (1997), two errors were possible. The first was to reason that each rabbit in the two groups would have the same number of sweets to eat, even if there were more rabbits in one group. The second possible error was to reason that the group which contained more rabbits would get more food (even though the number of sweets was the same). The first type of error would suggest the focus is caused by the quantity itself, that in the sharing of sweets it is only the sweets that are important, while the second type of error would suggest that this focus is caused by applying a direct reasoning strategy to a problem that requires inverse reasoning (the more rabbits = the more sweets). Correa et al. found that 5- and 6 year old children did not make a consistent pattern of errors on inverse relations problems. When they looked at the errors of 7 year olds though, the majority of errors were down to the over-application of the direct relation. This suggests that the over-application of the direct relation may represent a necessary stage of development of inverse reasoning.

This finding was later replicated by Squire and Bryant (2003) who showed that the difficulty of inverse relations was unaffected by two task variations; the first variation, related to Piaget's previous reports, examined whether increasing the size of the difference in the varied quantity (carrots) helped children think about the inverse relation. The second variation examined whether presenting problems in pictorial format rather than numerical format helped children to think about inverse relations.

There are three lessons about relational thinking in extensive quantity contexts that can be drawn from the sharing literature. The first is that inverse relations problems were shown to be consistently more difficult than direct relations problems. The second is that inverse relations can be understood more easily when the number of dimensions to think about is reduced. Whether this means the number of relations children need to attend to is enough to explain the difficulty with inverse relations forms the basis of this thesis and is considered in more detail when dealing with problem classification. The third lesson is about what makes inverse relations difficult to understand. The studies by Correa, Nunes, and Bryant (1998) and Squire and Bryant (2003) have shown that young children are quite erratic in their reasoning about inverse relations, while children on the cusp of understanding inverse relations with relations between quantities start to make more systematic errors. It is this question of what makes the inverse relation difficult that is considered next.

2.2.3 The Difficulty of Inverse Relations

Having presented examples of extensive and intensive quantity problems in which children have difficulties working with inverse relations, it is important to consider the reasons suggested by psychologists for this difficulty.

Stavy and Tirosh (2000) have suggested that a series of intuitive rules underpins the performance of children and adults on many scientific tasks. Their ideas build on earlier attempts by Resnick and Singer (1993) to describe children's intuitive thinking through their theory of protoquantitive schemas. It must be noted that Stavy and Tirosh's approach is considered here as it was developed to explain how children's knowledge of science develops. For comparison problems that cover the extensive and intensive quantity problems currently under investigation two intuitive rules were proposed: 'The more A, the more B' and 'The same A, the same B'.

Stavy and Tirosh (2000) argue that although these rules influence the thinking of both children and adults, younger children up to around the age of 8 years rely on intuitive rules across a wider variety of tasks than older children. At the heart of Stavy and Tirosh's theory is the idea that children interpret a novel problem with a known relation. As younger children often reason on a basic perceptual level, Stavy and Tirosh argue that this leads them to a greater proportion of 'the more A, the more B' judgements. Stavy and Tirosh illustrated this by looking at children's judgements about the equality of angles. They presented children (5–15 years) with two identical angles for which the length of the lines forming the angles differed (an irrelevant variation). This irrelevant variation influenced the thinking of children up to 12 years who judged the angles based on line length.

The proportional reasoning literature also reports children's susceptibility to irrelevant perceptual variations. For example, Levin (1979) asked children (4–7 years) to judge which of two lights burned for a longer period. Levin systematically varied three dimensions of the problem, two relevant to the solution (the time at which lights were switched on or off) and one irrelevant dimension (the intensity of the light). Levin found that children viewed light intensity as an indicator of burning time. The error pattern of this misconception reflected a 'the more A, the more B' intuitive rule with higher light intensity equating to 'longer duration' judgements.

Stavy and Tirosh (2000) argue that as children get older they are still influenced by intuitive rules though rule selection is based on the extrapolation of previously acquired logical schemes and not the perceptual variations observed with younger children. Stavy and Tirosh highlight proportionality and conservation as logical schemes that lead older children's frequent use of the rule 'same A, same B'.

The over-application of proportionality in comparative judgements of probability was demonstrated by Fischbein and Schnarch (1997). They asked children and adolescents (10–17 years) to compare the probabilities of two theoretical sets of coin tosses. Participants judged whether the probability of getting heads twice in the tossing of three coins was greater than, less than, or equal to, the probability of getting heads at least 200 times in 300 coin tosses. The majority of participants at each age level incorrectly treated the problem as one of proportional equivalence, stating that equal proportions mean equal probabilities.

The important point from Stavy and Tirosh's (2000) theory is that intuitive rules should predict the performance of children on a variety of tasks. As regards the current thesis, it is important to ask whether Stavy and Tirosh's intuitive rules can sufficiently predict and explain the difficulties children have when solving comparisons problems with extensive or intensive quantities. If Stavy and Tirosh's proposal that much intuitive reasoning by younger children is based on perceptual increases is correct, then it follows that direct relational comparisons should be made earlier than inverse relational comparisons regardless of whether extensive or intensive quantities are involved. Indeed much of the literature reviewed thus far comparing direct and inverse relations in either intensive quantity (Strauss and Stavy, 1982; Accredo, Adams and Schmidt, 1984; Desli, 1999) or extensive quantity (Kun, 1977; Suber, 1980; Sophian, Garyantes and Chang, 1997; Correa, Nunes and Bryant, 1998; Squire and Bryant, 2003) situations supports this view.

However, the problem with Stavy and Tirosh's (2000) theory is that it is difficult to predict when children might favour one rule over another. Stavy and Tirosh's general point that it could be tied to age is too vague to make predictions with. For example earlier work carried out by Stavy (Strauss and Stavy, 1982) showed with sweetness children solved direct relations items (controlled water, varied sugar), at a younger age than they were able to solve inverse relations items (varied water, controlled sugar). This finding is compatible with the intuitive rules theory. However, the analysis of judgements on failed inverse items showed children did not reason on the basis of perceptual differences (more water, more sweet). Instead, children tended to say the drinks would taste the same because the same amount of sugar was in each glass. Although Stavy and Tirosh could argue that this demonstrates the second intuitive rule (same a, same b) their theory fails to explain why perceptual differences were ignored by young children in this case. Even within the extensive quantities literature presented (Sophian, Garyantes and Chang, 1997; Correa, Nunes and Bryant, 1998), young children were shown to make inconsistent judgements on sharing of quantities problems. As Stavy and Tirosh's work was based on a general understanding of mathematics and science, it could be that their theory of intuitive rules is too simplistic to describe the development of children's thinking in intensive and extensive quantity situations.

Work by Squire and Bryant (2003), discussed in the previous section, argues that a child's 'same A, same B' or 'the more A, the more B' error on inverse relations problems indicates different levels of understanding. With their rabbits and carrots sharing problems they describe how at a lower level, children only saw the quantity of carrots as important in determining the size of the share. These children reasoned that the same number of carrots means the size of each share would be equal even though the number of rabbits varied. On the other hand, children with a higher level of understanding of the problem may understand that the number of rabbits is important to the solution of the problem, but be unable to think about the inverse relation, and so use direct relations to reach the a solution.

In summary, if children are using general intuitive rules then they should always understand direct relations earlier than inverse relations. The same pattern of results would also be expected with reasoning about the salient variable as with the sweetness problem. If children make errors on inverse relations problems with extensive and intensive quantities, this could indicate different levels of understanding of extensive and intensive quantity problems. Due to no systematic studies existing in the literature it is still unclear whether the type of quantities involved (extensive or intensive) have a significant effect on the capacity of children to solve relational thinking problems.

2.2.4 The Classification of Proportional Reasoning Problems

In addition to the notion of direct and inverse reasoning, another wave of research explored the idea that conceptual difficulty could not solely be explained by the proportionality schema described by Piaget. These researchers found that the type of proportional reasoning problem may also be important. This position is summarised by Lesh, Post and Behr (1988):

"In general, according to Piaget, adolescents' proportional reasoning develops from (1) a global compensatory strategy (often additive in nature) to (2) a multiplicative strategy without generalisation to all cases to (3) a final formulation of a law of proportions. However, in attempts to verify Piaget's theory, it has been noted that the level of reasoning that a child uses is often not consistent across tasks or even within a given task (e.g., when the number relationships or perceptual distracters or contextual variables are changed slightly). Even though the stages that Piaget describes have proven quite robust for describing children's behaviours on a given task, variability across tasks is often quite large." (p.104)

The observation of differences between tasks has led psychologists to think about how the underlying structure of the problem contributes to its difficulty and also about what criteria should be applied to distinguish problems from one another. The approach to this question has been considered in three general ways; the classification of problems on the basis of task variables, children's scholastic experience, and underlying mathematical structures.

2.2.4.1 The Classification of Multiplicative Problems by Task Variables

Tourniaire and Pulos (1985) examined the literature for clues and presented a detailed review covering many areas. This led to the production of a system of problem classification based primarily on task variables. Tourniaire and Pulos's classification suggests three factors affect children's performance on proportional reasoning tasks. The first factor influencing problem difficulty was the use of manipulatives, with Tourniaire and Pulos reporting that the use of manipulatives helped low-ability children solve proportional reasoning problems. A second significant factor was the familiarity of the problem context. This was in response to the criticisms of the Piagetian research, which they saw as using unfamiliar and complex tasks. On this question Tourniaire and Pulos (1985) report that:

"It (was) beneficial only if the subject was familiar with using ratios in that context." (p.190)

This raises another important question, which needs further consideration in the third section of the literature review, namely that of how much teaching there currently is on proportions in intensive and extensive quantity settings, and what impact this teaching might have on children's reasoning.

The third factor Tourniaire and Pulos (1985) considered was the classification of problem difficulty by problem type. They proposed that problem type was determined by two factors. Firstly, they argued that problems presented using mixtures are more difficult than those which do not involve mixtures. Secondly, that problems using continuous quantities are more difficult that those which use discrete quantities. The general argument was that continuous quantities are more difficult to work with, particularly for less able children. Tourniaire and Pulos also suggest that as many mixture problems are presented as continuous quantities this may further explain why children find them difficult.

As Tourniaire and Pulos's classification system was based on general observations of task variables within the literature, their system did not have a strong theoretical basis. This led them to over-generalise when explaining why certain task variables cause children problems. For example, when explaining why mixtures represent a distinct class of problem they state:

"First the elements of the ratio in a mixture problem constitute a new object e.g. red and yellow paint mixed makes orange, or a modified object, e.g., orange juice mixture with water makes a weaker orange juice. By contrast, no new object emerges in rate problems. Second, mixture problems require that the subject understands what happens when the two

elements are mixed. Third, in most mixture problems, the quantities are expressed in the same units, e.g. ounces, whereas in most rate problems the quantities involved are in different units, e.g. ounces and dollars. Dealing with quantities expressed in the same unit may be more confusing." (Tourniaire and Pulos, 1985, pp.183–84)

This explanation overstates the uniqueness of problems involving mixtures, ignoring the similarities that exist between them and other proportional reasoning problems. Harel, Behr, Post and Lesh (1991) point out that:

"the ratio elements in rate problems, as in mixture problems, do constitute a new object; the quantity of speed for example is a new quantity which emerges from the quantities of time and distance. Second, both rate problems and mixture problems require that the subject understands the physical situation when two quantities are combined. This is true whether this is a mixture of water and orange concentrate or a comparison between time and distance. Third, why should problems in which the quantities are represented in the same unit be more difficult than those in which the quantities are expressed in different units?"(p.127)

Although Tourniaire and Pulos (1985) highlight possible task variables which influence proportional reasoning as a system of classification, too many contradictions exist for it to be regarded as complete.

2.2.4.2 The Classification of Multiplicative Problems by Scholastic Experience

Around the same time as Tourniaire and Pulos' work, a different method was being used to classify proportional reasoning problems. Heller, Ahlgren, Post, Behr and Lesh (1989) attempted to classify proportional reasoning problems by looking at the types of problems used in textbooks in the United States. The Heller et al. classification system was concerned with the contexts in which children are exposed to proportional reasoning problems. This classification system was based on two broad characteristics. The first characteristic was 'problem setting'; this referred to what would be regarded as the surface features of the problem. Heller et al. argued that surface features vary in their degree of familiarity to children. The problem setting consists of the objects included in the problem; for example, speed problems can be posed using different objects, e.g. a car driving down a road, or a person running around a track. Problem setting also consists of the variables used to describe the objects (e.g. distance, time) and the units of measurement used (e.g. miles, metres, hours, minutes).

The second characteristic of the proportional reasoning context proposed by Heller et al. (1989) was the ratio type. Ratio type refers to the way in which textbooks represent different problems with ratios. Nine ratio types were identified. For ease of reading they are displayed in Table 2.1.

Table 2.1 Heller et al.'s (1989) Classification of Ratio Problems

Ratio type	Example
Distribution: divide something equally among people, groups or objects	Cookies per person
Packing: spread something evenly over a spatial dimension	Books per foot of shelf space
Package size: count or measure	Candies per box
Exchange: trading one kind of thing for another	Money per week
Mixture	Taste
Speed	Miles per hour
Consumption/Production: how efficiently something is consumed	Gallons of oil burned by a furnace per hour
Scaling	Inches per mile map scale
Conversion	Pounds per dollar

Unlike Tourniaire and Pulos (1985), Heller et al. (1989) looked directly at how problem context influenced children's success using both comparison and missing value problems. They considered two problem settings (familiar vs. unfamiliar) and three of their nine identified ratio types (speed, exchange and consumption). In total they tested 254 children aged 11–12 years using paper and pencil story problems. Heller et al. analysed separately the impact of problem context on comparison and missing value items. For comparison problems a significant interaction was found between problem setting and ratio type. The basic interpretation of this was that in less familiar settings the difficulty increased as the ratio type changed with exchange the easiest, followed by speed, then consumption. For missing value problems there was no interaction of ratio type by familiarly of setting but a general increase in difficulty by ratio type.

Heller et al. (1989) concluded that ratio type was more important than problem setting. The first problem with the Heller et al. classification system is that it is difficult to understand what is meant by familiar and unfamiliar problem contexts. For example, while discussing speed problems they explain that a familiar context would be a person running around a track, with an unfamiliar context being a car driving down a road. It is difficult to understand why the second problem would be an unfamiliar context. To a child that enjoys motor racing this could be just as familiar a context. It is argued therefore that as proportional reasoning problems include many situations which children encounter on a daily basis, it is difficult to establish how unfamiliar a context is. For the current thesis, the more important point is whether particular situations are familiar to children. The only clue to this is to look at the collective experience of children through their school work. The issue of what proportional reasoning situations are taught to children will be presented in the third section of this literature review.

The second problem with the Heller et al. (1989) classification system was that although they tried to identify different types of ratios they considered only surface features which may only be described as task variables. Surface features reveal little about why a problem should be more or less difficult for children. Heller et al. do not suggest why their ratio types were different in terms of difficulty, and their empirical evidence did not test all the ratio types they identified. It is questionable whether, although the classification is based on contexts presented in textbooks, the underlying problem structures are distinct enough to warrant separate classifications.

2.2.4.3 The Classification of Problem by Multiplicative Structure

The question of how best to classify multiplicative problems by their underlying structure resulted in two competing theories (Vergnaud, 1983 and Schwartz, 1988). As these theories were described in chapter 2 (see page 8) only the main points are presented here.

Of interest to the current thesis are the multiplicative structures which involve more than two quantities. For these situations, Vergnaud (1983) created the distinction between product of measures and multiple proportion situations. Products of measures situations involve the multiplication of two measures to produce a third measure, which is neither of the initial measures. As this third quantity according to Vergnaud, is a product it is always directly proportional to the referent quantities. A well-known example of a product of measures situation that children meet is the Cartesian product problem. A typical Cartesian product would involve the multiplication of a set number of shirts by a set number of shorts to produce a number of possible outfits.

An important point made by Vergnaud regarding product of measures problems is that the product remains an elementary (or extensive) quantity. That is a quantity that is still susceptible to addition (1 outfit + 1 outfit = 2 outfits).

The multiple proportions structure is quite similar to the product of measures structure in the sense that in its most basic form it involves relations between three elementary (or extensive) quantities. For multiple proportion problems, though each of the quantities in the problem remains an independent quantity.

An example of a multiple proportions problem would be the calculation of food consumption by a set number of people over a specified period. Vergnaud specifies that each quantity within the multiple proportions structure (kilograms of food, number of people and length of time), though related in the problem setting, remains an independent magnitude.

The problem with Vergnaud's theory is to find a place for intensive quantity problems. The simplest forms of intensive quantities involve two independent measures the quotient of which produces the intensive quantity e.g. (speed = distance/time). This differs from Vergnaud's product of measures structure in two ways. Firstly the outcome quantity in a product of measures problem is reached by a multiplication, the outcome quantity in an intensive quantity problem is reached through division. This leads to the second difference; within Vergnaud's product of measures structure; the product is always directly proportional to the referent quantities. Intensive quantities can be both directly and inversely proportional to the referent quantities, depending on the problem posed. Vergnaud's descriptions of the multiple proportions structure also fails to capture the intensive quantity structure as multiple proportions involve independent quantities, while an intensive quantity is dependent on both the referent quantities.

Schwartz (1988) picked up on this problem and expressed a different idea about the classification of multiplication problems with his theory of referent transforming operations. Schwartz's argument is that the link between numbers and referents underpins important conceptual distinctions between multiplicative and additive structures. Viewing multiplication situations in terms of referents requires a distinction between extensive and intensive quantities not made by Vergnaud.

Like Vergnaud, Schwartz's theory of referent transforming operations leads to three types of multiplicative structures. The majority of situations faced by children involve the triad of one intensive and two extensive quantities (I E E'). This structure incorporates many of the problems Vergnaud regarded as isomorphism of measures. Schwartz also distinguished between a triad of extensive quantities E E' E', which corresponds closely to Vergnaud's product of measures situations, and a triad of intensive quantities I I' I' that could be regarded as complex multiple proportion situations.

According to Schwartz (1988), the majority of multiplicative problems encountered by children are of the I E E' structure and should be introduced as such for children to think about the relations between referents. The problem with Schwartz's theory is that it is more general than Vergnaud's and as such it does not capture some important structural differences identified by Vergnaud. For example Vergnaud (1983) presented a study by Vergnaud, Ricco, Rouchier, Marthe and Metregiste (1978) in which they tested whether the distinction between product of measures and multiple proportion items results in any observable differences in difficulty. They asked a sample of 84 children aged 11-12 years to solve a series of product of measures and multiple proportion problems. The number of quantities involved for each problem type was controlled. For product of measures situations, children calculated area, while on multiple proportion problems children had to calculate the quantity of milk produced by x cows at a rate of y litres per cow per day over a set period of time. Children found the product of measures problems more difficult than multiple proportion problems. This shows that the difference between product of measures problems and multiple proportions problems represents a conceptual difference. In Schwartz's theory this potentially important distinction is lost with both product of measures and multiple proportions coming under his E E' E' classification.

2.2.4.4 Resolving the Schwartz and Vergnaud Debate.

Although Vergnaud does not distinguish between extensive and intensive quantities, his theory allows some predictions to be made about the relative difficulty of intensive quantity problems. When intensive quantities are compared or computed Vergnaud (1983) refers to this as a comparison of functions for which *'the function is also a variable'* (p.166). For Vergnaud this problem becomes one of multiple proportions. This classification contradicts Vergnaud's earlier definition of multiple proportion structures comprising three or more independent quantities. It does however provide a chance to test Schwartz's (1988) argument that the distinction between extensive and intensive quantities is significant. If Vergnaud is correct and problems involving the comparison of intensive quantities are conceptually equivalent to general multiple proportions problems involving three independent (or extensive) quantities, then the multiple proportions problem structure should determine the difficulty. And if this is the case, then it can be expected that no significant differences will exist between multiple proportion problems involving only extensive quantities or those involving intensive quantities. On the other hand, If Schwartz's (1988) idea about the importance of referents is correct, then multiple proportion problems involving only extensive quantities would be classified under his E E' E' structure with those involving intensive quantities coming under his I E' E' structure. Therefore, one would expect that significant differences would be observed in the relative difficulty of problems. This distinction has yet to be tested as Schwartz failed to present empirical evidence for his theory. Currently no systematic studies of relational thinking exist in the proportional reasoning literature which considers whether this distinction between extensive and intensive quantities is conceptually important.

2.2.5 Evidence from the Literature of the Need to Distinguish Between Extensive and Intensive Quantities

The most encouraging evidence that the distinction between extensive and intensive quantity product of measures problems is an important one, comes from studies in which the intention was *not* to make such a comparison. For example, when looking at children's solution strategies to proportional reasoning problems, Resnick and Singer (1993) report on the success of children solving problems in their own investigations and those of an earlier study by Ricco (1982). Resnick and Singer's example of a problem given to 7 and 8 year old children is:

"Raymond reads 2 pages of his book everyday. George reads 4 pages of his book. When Raymond has finished 20 pages, how many pages will George have finished?" (Resnick and Singer, 1993, p.120)

Resnick and Singer reported children succeeded in solving this problem solving by filling out tables comprising three columns: the first relating to days, the second to Raymond's pages, and the third to George's pages. Children generally used a building up strategy (Hart, 1984). That is for every day they added 2 to Raymond's column and 4 to George's, thus relating each quantity to days. Resnick and Singer then describe a similar problem used earlier by Ricco (1982). Ricco gave children aged 7–11 years problems in which the prices of a certain number of pens had to be calculated when information was given about other multiple pen purchases. As with Resnick and Singer's problem, the data was presented to children in tabular form, but with only two columns, the first column representing the number of pens and the second column the price paid. Under these conditions, the children found it more difficult to solve the problem, with the building-up solution observed by Resnick and Singer only observed in 9 year olds.

As Resnick and Singer (1993) used these problems to argue for an additive basis to children's solutions to multiplicative problems, they do not consider why two seemingly similar problems differ in levels of difficulty. Also, as the studies were conducted independently and factors such as the numbers used in the problem differed between studies it is difficult to provide a definitive reason for the variation in performance. Despite these issues, it is not unreasonable to assume that the underlying multiplicative structures of these problems influenced their respective difficulty levels. Resnick and Singer's easier item involved children establishing relations between three extensive quantities (Raymond's books, George's books, days), while Ricco's (1982) problem involved the relations between two extensive quantities (money, pens) and an intensive quantity (cost per pen).

Further evidence of the need to consider the distinction between extensive and intensive quantities systematically is found in Kaput and West's (1994) reports of children's solutions to missing value proportional reasoning problems prior to formal instruction. Kaput and West were interested in seeing how different missing value problems promote different types of informal reasoning strategies.

They identified informal reasoning patterns for solving missing value proportional reasoning problems and discussed how different problems might facilitate building up or unit factor strategies. The building up strategy has been mentioned earlier in relation to Resnick and Singer's (1993) investigation; a unit factor approach is similar to Vergnaud's (1983) functional solution in as much as it would involve calculating the cost of 1 car, then multiplying that by the total number of cars required. Kaput and West (1994) predicted that with missing value problems children would be more likely to use a building up strategy if they contained a '*semantic holder*' as demonstrated by the quantity '*placemats'* in the following problem:

"A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silverware in its table settings last night, how many pieces of china did it use?" (Kaput and West, 1994, p.245)

Kaput and West also predicted that problems without this semantic holder such as their Italian dressing problem would be solved using a unit factor approach and would therefore be more difficult:

"To make Italian dressing you need four parts vinegar for nine parts oil. How much oil do you need for 828 ounces of vinegar?" (Kaput and West, 1994, p.245)

Kaput and West gave these problems to 138 children aged 11–12 years. In line with their predictions, the placemats problem proved significantly easier than the Italian dressing problem. In explaining why the placemats problem was easier they suggest that the inclusion of containment made the problem significantly easier:

"A problem is more likely to be approached with a build-up strategy if the given unit quantities are associated in a situation of containment, for example the placemats problem, where the placemats hold the two sets of items together. This facilitates unit formation at the second, units of units, level." (Kaput and West, 1994, p.272)

The Italian dressing problem though, is seen as difficult because the extensive quantities lost their identity in the mixture.

Once again there are two problems with different underlying mathematical structures: the placemats problem involved relations between three extensive quantities (china, silverware, placemats), and the Italian dressing problem involved relations between two extensive quantities which formed an intensive quantity (vinegar, oil, salad dressing). As in Resnick and Singer's (1993) study, the extensive quantity problem was easier to solve than the intensive quantity problem. It is also suggested by Kaput and West that these problems differ in terms of difficulty as children approach them with different strategies. In their study, Kaput and West did not distinguish between the placemats and the Italian dressing problem on the basis of extensive and intensive quantities.

Of interest to the current thesis is the fact that the easier reading problem (Resnick and Singer, 1993) and placemats problem (Katput and West, 1994), are essentially those involving the relations between three extensive quantities, while the purchase of pens problem (Ricco, 1982) and the Italian dressing problem (Kaput and West, 1994) involved two extensive quantities which form an intensive quantity. Similar examples of children using a third extensive quantity to solve proportional reasoning problems can be found in Lamon (1993, 1994). As none of the studies quoted aimed to systematically compare extensive with intensive quantities, caution must be taken with the interpretation of the result.

Although this literature does not allow definite conclusions to be drawn about the distinction between extensive and intensive quantity problems, there are enough relevant findings to suggest that more work is needed in this area.

2.3 Section 3: Teaching Practice and Intensive Quantity Understanding

This section looks at the extent of children's school-based experience with intensive quantities. Looking at children's current learning opportunities will help establish what formal instruction children receive on both intensive quantities and inverse relations. This section of the literature review will concentrate on the teaching of children up to Year 4 (8–9 years) as this is the time when studies (Piaget, 1970; Piaget and Inhelder, 1974; Strauss and Stavy, 1982; Desli, 1999) have shown that inverse relations with intensive quantities can first be understood.

At the time of data collection for the current thesis, children in England were working from the National Numeracy Strategy (DfEE, 1999). It is on this that children's learning experiences will have been based. An additional publication, *Mathematical Vocabulary* (DfEE, 2000), will also be considered as this was produced to help fulfil the goals of the National Numeracy Strategy. *Mathematical Vocabulary* will provide insights into any intensive quantity related terms which children would be expected to know. This section finishes by looking at the recent revisions to the National Numeracy Strategy (DfES, 2003) to see whether this has led to significant changes in the teaching of intensive quantities.

2.3.1 Intensive Quantities, Inverse Relations and the National Numeracy Strategy

The review of the National Numeracy Strategy's teaching guidelines and goals is presented as follows. Firstly, the general goals and key strategies related to the current thesis are identified: inverse relations and intensive quantities. Secondly, information is presented on the expected vocabulary to be mastered. Thirdly, the relevant areas of the suggested teaching programme and example problems are presented; this section is presented by year group. It is important to note that the data collection for the studies reported in the current thesis took place during the summer term, therefore it would be expected that the teaching aims identified by year would have been covered for the year groups tested.

One of the key objectives set out in the National Numeracy Strategy is to use primary mathematics as a basis for the foundations for algebra. It is under this objective that ideas important to the current thesis are laid out. The first is that children must understand inverses:

*"**Using inverses:** Another important idea in both number and algebra is the use of an inverse to 'reverse' the effect of an operation. For example, the inverse of doubling is halving, of adding 7 is subtracting 7, and of multiplying by 6 is dividing by 6. Once they have grasped this idea, pupils can use their knowledge of an addition fact such as 4 + 7 = 11 to state a corresponding subtraction fact: 11 – 7 = 4. Similarly, pupils should be able to use their knowledge of a multiplication fact such as 9 × 6 = 54 to derive quickly a corresponding division fact: 54 ÷ 6 = 9."* (DfEE, 1999, Section 1, p.9)

It is quite clear from this that the emphasis of understanding inverses is a specific idea of 'inversion', related to inverse relations between numbers but not quantities. It must be remembered that understanding inverse relations between quantities is an important idea in intensive quantity development. The second point from the section on laying down the foundations for algebra relevant to the current thesis is 'expressing relationships':

*"**Expressing relationships:** When discussing graphs drawn, say in science, ask children to describe in their own words the relationships revealed: for instance, 'every time we added another 20 grams the length of the elastic increased by 6 centimetres'. They can also be asked to use and make their own simple word equations to express relationships such as: Cost = number × price."* (DfEE, 1999, Section 1, p.9)

Here there is explicit reference to understanding relations between quantities, though it must be noted that the statement contains some confusions. The statement 'cost = number

× price' is misleading in assuming that cost always involves direct relations. The correct expression of this formula would be: Total cost = Number of items × Price per item (assuming that only one type of item is being purchased). Cost, which is an intensive quantity, is more accurately described as Cost = Number of items ÷ Money paid. This intensive interpretation of cost is one which children need to acquire to enable them to do things like compare products in supermarkets. This may seem like a trivial point, but as will be shown, introducing this confusion from the start restricts the experiences of cost recommended by the National Numeracy Strategy.

The only other references relevant to the concept of intensive quantity set out by the National Numeracy Strategy appear during the identification of the framework's five strands. Under the heading of 'numbers and the number system' it is seen as important to understand:

"5: Fractions, decimals and percentages, and their equivalence: ratio and proportion." (DfEE, 1999, Section 1, p.39)

While under 'solving problems' it is important to be:

"3: solving problems involving numbers in context: 'real life', money, measures." (DfEE, 1999, Section 1, p.39)

Although the general strategic aims of the National Numeracy Strategy do not mention intensive quantities explicitly, some important ideas related to intensive quantities appear. However, in the case of using inverses, only numerical relations and not relations between quantities are covered. Also, when relations between quantities are mentioned, these are unfortunately presented as though these relations are always direct. As the intensive quantity of cost is mentioned specifically, it is important to look at how teachers are instructed to teach this concept in order that one may know what children might be expected to understand in the studies conducted as part of this thesis.

2.3.2.1 Mathematical Vocabulary

The National Numeracy Strategy supplementary publication, *Mathematical Vocabulary*, lists the terms deemed important to mathematical development:

"The purpose of this book is to identify the words and phrases that children need to understand and use if they are to make good progress in mathematics. It is designed to support the National Numeracy Strategy." (DfEE, 2000, p.1)

In relation to the current thesis, terms related to two intensive quantities are given. The first is in relation to speed and is presented under the heading of 'time', while the second list of terms are important to the concept of cost and are given under the general heading of 'real life and money'. Displayed below in Table 2.2 are the terms related to speed and cost which it is expected that children in years Reception to Year 4 will understand.

Table 2.2 Terms Children Should Understand Relating to Speed and Cost as Part of the National Numeracy Strategy (DfEE, 2000)

Year Group	Terms related to speed	Terms related to cost
Reception	*Quick, quicker, quickest, quickly, slow, slower, slowest, slowly.* (p.9)	*Price, cost, dear, costs more, cheap, costs less, cheaper, costs the same as.* (p.9)
Year 1	*Quick, quicker, quickest, quickly, fast, faster, fastest, slow, slower, slowest, slowly.* (p.14)	*Price, cost, dear, costs more, cheap, costs less, cheaper.* (p.13)
Year 2	*Quick, quicker, quickest, quickly, fast, faster, fastest, slow, slower, slowest, slowly.* (p.17)	*Price, cost, dear, costs more, cheap, costs less, cheaper.* (p.17)
Year 3	*Quick, quicker, quickest, quickly, fast, faster, fastest, slow, slower, slowest, slowly.* (p.22)	*Price cost, dear, costs more, more/most expensive, cheap, costs less, cheaper, less/least expensive, value, worth.* (p.21)
Year 4	*Quick, quicker, quickest, quickly, fast, faster, fastest, slow, slower, slowest, slowly.* (p.26)	*Price, cost, dear, costs more, more/most expensive, cheap, costs less, cheaper, less/least expensive, value, worth* (p.25)

From an early age, children are expected to be familiar with terms essential to the understanding of speed as an intensive quantity. Although the number of terms to be understood increases between Reception and Year 1 to include more terms related to speed, words such as 'quickest' and 'slowest' appear in the strategy from the start. It is not unreasonable therefore to assume that by Year 4, children will have had significant experience of these terms. More interesting is the progression of vocabulary recommended to develop the idea of cost. Although terms such as 'cheap' are emphasised very early on, more explicit reference is made to the idea of cost as an intensive quantity from Year 3 onwards when the terms 'value' and 'worth' are introduced. Understanding such terms requires consideration of the relation between an item to be purchased and the price to be paid. To understand how these ideas are promoted though the National Numeracy Strategy it is important to give an account of both recommended teaching plans and sample problems. These will be reported by year group to give an overall picture of the expected progression.

2.3.2.2 Reception

Although the National Numeracy Strategy's *Mathematical Vocabulary* suggests that children should understand terms such as 'quickly' and 'slowly', the strategy does not provide any specific suggestions as to how Reception class teachers should broach the subject. Ideas related to cost do receive more attention with the suggested teaching programme under the heading of 'solving problems' stating:

" *Use developing mathematical ideas and methods to solve practical problems involving counting and comparing in a real or role play context. Begin to understand and use the vocabulary related to money. Sort coins, including £1 and £2 coins, and use them in role play to pay and give change."* (DfEE, 1999, Section 3, p.2)

As the emphasis is purely on the understanding of coins and coin values, it is perhaps not surprising that direct price comparisons receive all the conceptual attention, while product as a quantity is used merely to vary the problem setting. This is highlighted by the sample problems provided:

"Discuss (unpriced, later priced) items in the classroom 'shop'. Say which might cost more, or which might cost less. For example, respond to: An apple costs 4p. An orange costs 1p more. What does the orange cost?" (DfEE, 1999, Section 4, p.11)

"Chews cost 2p each. How much do 2 chews cost?" (DfEE, 1999,Section 4, p.21)

2.3.2.3 Year 1

As children move to Year 1 the emphasis of the National Numeracy Strategy changes very little. In relation to the understanding of speed, general guidelines do now appear:

"Understand and use in context, comparatives such as faster, slower, takes longer." (DfEE, 1999, Section 5, p.78)

No specific examples are provided in the strategy on how best to introduce children to comparisons involving speed. With cost, the emphasis remains on coins, coin value and coin comparison. Within the teaching programme teachers are advised that children should:

"Work out how to pay an exact sum using smaller coins." (DfEE, 1999, Section 3, p.7)

Again, sample problems all emphasise multiple purchases at a fixed price:

"A plum costs 5p. Find the cost of three plums, using coins if necessary." (DfEE, 1999, Section 5, p.26)

Or the comparison of different products with different prices:

"An apple costs 7p. An orange costs 10p more. What does the orange cost?" (DfEE, 1999, Section 5 p.12)

Thus far, teachers are being recommended to teach the direct relation between total costs when products are controlled.

2.3.2.4 Year 2

It is during Year 2 that the first stages towards building a concept of speed receive attention, other than the identification of terms. In Year 2, it is advised that children:

"Make estimates and check using a simple timer in PE, science ... or at home. For example: Estimate then check: who takes least/most time to hop across the hall." (DfEE, 1999, Section 5, p.79)

This is interesting, as comparing the time of travelling a fixed distance promotes the inverse relation between speed and time, though this is not referred to explicitly. For cost, the same direct relations and understanding of coin value are recommended and as such do not require further examples.

2.3.2.5 Year 3

For Year 3, the introduction of speed as an inverse relation, with time taken when distance travelled is controlled, is reinforced:

"Make estimates and check using a simple timer in PE, science ... or at home. For example: Estimate, then check: the greatest and the least time taken to run 200m." (DfEE, 1999, Section 5, p.79)

When looking at cost, it will be remembered that in terms of mathematical vocabulary children in Year 3 are expected to be introduced to terms explicitly relating to cost as an intensive quantity (worth, value). It is also at this age that children are expected to give more thought to the idea of cost by explaining problem solutions:

"66–71 solve word problems involving numbers in "real life", money and measures, using one or more steps, including finding totals and giving change, and working out which coins to pay. Explain how the problem was solved." (DfEE, 1999, Section 3, p.15)

Unfortunately, in spite of this new vocabulary which should promote cost as an intensive quantity, the sample problems given to Year 3 teachers as a guide still only tackle direct relations between price and cost, disregarding the significance of the product purchased:

"It costs 75p for a child to swim. How much does it cost for two children?"

"Dad bought three packets of cornflakes at 70p each. What was his change from £3?" (DfEE, 1999, Section 5, p.69)

In short, no specific instructions are given on how ideas such as value or worth should be taught or introduced.

2.3.2.6 Year 4

The recommended teaching programme for Year 4 teachers does not develop any further ideas relating to speed, although the idea of races is made explicit:

"Use a stop watch or other timers to measure and compare times of events: for example, use a stopwatch in PE to time races." (DfEE, 1999, Section 6, p.98)

For cost, the focus of teaching remains solely on direct relations, though it must be noted that relations between quantities problems do gain more prominence, as the idea of proportions is first mentioned explicitly for this year group:

"Point 26 begin to use ideas of simple proportions: for example, "one for every..." and "one in every..."." (DfEE, 1999, Section 2, pp.18–19)

At this age, children are expected to base this around the idea of area (a product of measure problem which only exists with direct relations), while beginning to think about direct relations between previously unrelated quantities:

"Use a computer program to solve number puzzles; for example, to fill a given number of carriages on a train with given numbers of people." (DfEE, 1999, Section 6, p.78)

"Solve story problems about numbers in real life, choosing the appropriate operation and method of calculation. To cook rice, you need 5 cups of water for every cup of rice. You cook 3 cups of rice. How many cups of water do you need?" (DfEE, 1999, Section 6, p.82)

2.3.2 Conclusions

The National Numeracy Strategy in operation at the time of the data collection for the current thesis does not teach explicitly the concept of intensive quantity, although two intensive quantities were promoted (cost and speed) with the former receiving a lot of attention under the banner of 'real life problems'. There is no teaching of the relations that exist between variables. This results in the promotion of teaching methods that view cost problems as always involving direct relations between price and cost, and speed problems as only involving inverse relations between finishing times of various races. The root of this problem appears to be a curriculum-based one promoting inverse relations solely as a question of number relations and not as a possibility for relations between quantities.

As the National Numeracy Strategy has recently undergone a series of revisions (Primary Framework for Literacy and Mathematics (Dfes, 2003), it is important to investigate whether these revisions have tackled the concept of intensive quantity.

2.3.3 Recent Revisions to the National Numeracy Strategy

In 2003 the National Numeracy Strategy was revised as the Primary Framework for Literacy and Mathematics (DfES, 2003). One of the key objectives behind this revision was to simplify the teaching and to focus on the key ideas for mathematical development. On this shift from quantity to perceived quality it is remarked:

"Another key difference is that the objectives in the 1999 framework for mathematics have been slimmed down to give a clearer sense of the important aspects of mathematics that need to be taught to children." (DfES, 2003)

In relation to the current discussion on the teaching of intensive quantities, this decision to reduce the content of the Numeracy Strategy without making any major conceptual changes to the teaching of numeracy means that there are now fewer references made to intensive quantity problems than in the previous strategy. Although inversion still appears as relations between numbers and not quantities, the same restricted view of proportional relations for Year 4 children as direct relations also remains:

"Use the vocabulary of ratio and proportion to describe the relationship between 2 quantities (e.g. "There are 2 red beads to every 3 blue beads, or 2 beads in every 5 beads are red") and estimate a proportion (e.g. "About one quarter of the apples in the box are green.")." (DfES, 2003)

The Primary Framework for Literacy and Mathematics tackles neither intensive quantities nor inverse relations between quantities. In the original Numeracy Strategy, terms relevant to intensive quantity understanding appeared primarily in the publication of expected mathematical vocabulary of children, *Mathematical Vocabulary*. This promoted language related to speed and cost as intensive quantities, although terms such as 'faster than' or 'more expensive' were recommended even in the vocabulary that Reception teachers were expected to use. The current Primary Framework for Literacy and Mathematics has completely omitted any reference to terms related to speed for children up to Year 4 (which are of interest to the current thesis). The terminology related to cost has been greatly reduced, with the level of expected vocabulary reaching its most extensive by Year 2, though only containing a few terms related to the idea of money other than cost:

"Coin, Pound, £, penny/pence, price, cost, pay, costs more/less, change, total, how much?" (DfES, 2003)

To conclude, although very little instruction was given and no explicit references are made to intensive quantities in the original numeracy strategy, there is much less in the Strategy's latest incarnation. Therefore, it is necessary to examine whether the concept of intensive quantities receives significant attention in other mathematics curricula.

2.3.4 The Teaching of Intensive Quantities

To assess the teaching practices of every country for evidence of intensive quantity teaching is beyond the scope of this thesis. However, it is important to show that learning about intensive quantities forms an important part of other teaching systems: the idea of intensive quantities and the teaching of this idea is not just an abstract concept used in academic debate but represents a meaningful classification of multiplicative problem.

In Japan, children receive explicit teaching of intensive quantities under the heading of 'per unit quantities' (Ishida, 2003). The teaching of intensive quantities begins during the English equivalent of Year 5 (9–10 years). With regard to the teaching of intensive quantities, the goals of the Japanese primary curriculum clearly state:

"*B. Quantities and Measurements*

(3) Children should understand how to compare and express quantities that can be understood as the ratio of two different quantities, and should make use of them.

(a) To be able to conceptualise the idea of per unit quantities.

(b) To understand the meaning of how to express velocity, to think about finding velocity, and to be able to find the velocity of something." (p.13)

Students in Japan receive a total of 10 lessons dedicated to the concept of intensive quantity which are divided into three teaching sections. The first section (spread across four lessons) is of most interest to the current thesis as it is focused on developing the general idea of per unit quantities. A range of intensive quantity problems is presented to children covering density, speed, cost, and harvest yields. The emphasis in these early lessons is to create a conceptual understanding of intensive quantities with the distinction between direct and inverse relations made from the start as shown by this excerpt from the first lesson in a Japanese mathematics curriculum approved textbook (Ishida, 2003):

"Which room is more crowded?

Room	*A*	*B*	*C*
No. of Tatami	*10*	*10*	*8*
No. of Pupil	*6*	*5*	*5*

Q1 Which room is more crowded A or B?

Q2 Which room is more crowded B or C?" (Ishida, 2003, p.64)

From this starting point children are given instruction on the computation of intensive quantities by comparing rooms A and C from the sample problem through the calculation and comparison of tatamis per pupil and pupils per tatami. The second and third sections of the Japanese curriculum on intensive quantities for this age group are based on developing children's concept of speed.

It can therefore be concluded that although the teaching of intensive quantities is not prominent in the English curriculum where the data collection for the thesis was conducted, it does appear as an important mathematical idea in other teaching cultures.

CHAPTER 3

3 STUDY 1: DIRECT AND INVERSE RELATIONS WITH THE INTENSIVE QUANTITY TASTE: METHODOLOGICAL CONSIDERATIONS AND LINKS TO THE GLOBAL CONCEPT OF INTENSIVE QUANTITY

3.0 Aim

The aim of Study 1 is to work with some of the distinctions created within the proportional reasoning literature to see how they influence performance on direct and inverse relations problems. This will allow the examination of two issues. The first issue regards the methodology to be employed throughout the thesis. The second issue regards the link between direct and inverse relations within the wider concept of intensive quantity. To address these aims, Study 1 is reported as two experiments: Experiment 1 tackles methodological considerations, while Experiment 2 covers the links between direct and inverse relations and the wider concept of intensive quantity. It is important to note that children from the same sample participated in both the experiments that comprise Study 1.

3.1 Experiment 1: Comparison of Problem Presentation

The aim of Experiment 1 is to compare children's performance on direct and inverse relations problems across three modes of presentation (physical demonstrations, computer diagrams and manipulatives). This experiment aims to investigate two particular comparisons. Firstly, whether a difference in performance is observed between intensive quantity problems which are presented by physical demonstration and those displayed as computer diagrams. The second comparison is whether the use of manipulatives results in better performance on direct and inverse relations problems.

Children observe intensive quantities in many day-to-day activities; for example, when mixing a drink, running a race, or purchasing goods. When conducting research, it is not always practically possible to replicate these scenarios in the classroom exactly as children experience them in their lives. Later studies in the thesis (Chapter 4) will make comparisons between extensive and intensive quantity problems. When deciding the best way to present a problem to children, it is important to establish whether the same level of understanding can be inferred from children's performance on intensive quantity problems displayed more practically as computer diagrams than when presented within a context closer to their everyday experiences. If significant differences are observed here it will raise the question of ecological validity and so would render it difficult to generalise results of comparisons between intensive and extensive quantities problems presented to the children using computer diagrams.

The second issue is the significance of presenting problems with manipulatives, as reported in Tourniaire and Pulos's (1985) review of the proportional reasoning literature, and whether this can be extended to relational thinking about intensive quantities? Tourniaire and Pulos reported that success in proportional reasoning by children of low ability was facilitated by the presentation of problems with manipulatives. However, Tourniaire and Pulos did not expand on this finding to suggest the types of proportional reasoning problems solved more easily with manipulatives. It will be of interest to find out whether providing children with manipulatives can facilitate performance on direct and inverse intensive quantity problems. These comparisons form the basis of Experiment 1.

Experiment 1 compares children's performance on direct and inverse relations across three modes of presentation (physical demonstration, computer diagrams and manipulatives). Two main comparisons are made during Experiment 1. The first is between intensive quantities as experienced in everyday life (physical demonstration) and those presented in the form of computer diagrams. Independent groups of children will receive direct and inverse relations problems observing either the mixing of actual drinks or computer diagrams. The second comparison is between problems presented with or without manipulatives. For this a third independent group of children will be introduced. The children will be given direct and inverse relations problems using manipulatives (coloured counters) to represent the liquids which are being mixed. The data from this third group is used to indicate whether Tourniaire and Pulos's (1985) finding that manipulatives aid proportional reasoning can be applied to direct and inverse relations problems.

3.1.2 Hypotheses

To make these comparisons, Experiment 1 will test two experimental hypotheses.

Hypothesis 1:

'Significant differences will be observed between children receiving physical demonstrations and those receiving computer diagrams on direct and inverse relations problems'

Hypothesis 2:

'Children receiving problems with the aid of manipulatives will perform significantly better than those children who do not on direct and inverse relations problems'

3.2 Methodology

3.2.1 Participants

A sample of 228 children (107 boys and 121 girls) from eight Oxfordshire nursery and primary schools participated in Experiment 1. The sample consisted of six different age groups as displayed below in Table 3.1.

Table 3.1 Participant Summary

Age group	No. of children	Age range (years:months)	Mean age *(SD)*
4 years	28	4:02–4:11	4:07 *(.03)*
5 years	45	5:03–5:11	5:08 *(.02)*
6 years	30	6:00–6:11	6:06 *(.04)*
7 years	30	7:00–7:11	7:06 *(.03)*
8 years	63	8:00–8:11	8:06 *(.03)*
9 years	32	9:00–9:09	9:04 *(.02)*

3.2.2 Design

Experiment 1 employs a mixed model 6×3×2 design: age (4 years vs. 5 years vs. 6 years vs. 7 years vs. 8 years vs. 9 years – between participants) by mode of presentation (physical demonstration vs. computer diagrams vs. manipulatives – between participants) by relation type (direct relations vs. inverse relations – within participants). A total of eight items incorporating two taste situations (sugar/water and sugar/lemon juice) comprised Experiment 1. Four of the items measured performance on direct relations when different quantities were controlled. Four items measured performance on inverse relations when different quantities were controlled (See Table 3.2).

Table 3.2 Problems Used in Experiment 1

Relation	Quantity controlled	Quantity varied	Intensive quantity expression
Direct	Water	Sugar	Sweetness
	Sugar	Water	Wateriness
	Lemon	Sugar	Sweetness
	Sugar	Lemon	Sourness/Sharpness*
Inverse	Water	Sugar	Wateriness
	Sugar	Water	Sweetness
	Lemon	Sugar	Sourness/Sharpness*
	Sugar	Lemon	Sweetness

* The language used was determined by the child's report of what lemon tasted like

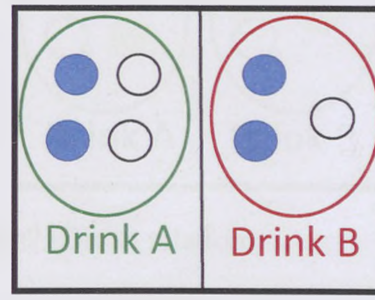
3.2.2.1 Mode of Presentation

The current study tested three independent groups on understanding direct and inverse relations. Each group received the problem information in a different form (physical demonstration, computer diagrams or manipulatives. Figure 3.1 (a–d) shows the computer diagrams and manipulatives versions of the items. For the physical demonstration condition the problems mirrored those shown as computer diagrams, the only difference being that children actually saw the sugar dissolve in the water/lemon solution.

a) Water controlled/sugar varied

Computer diagrams

Manipulatives



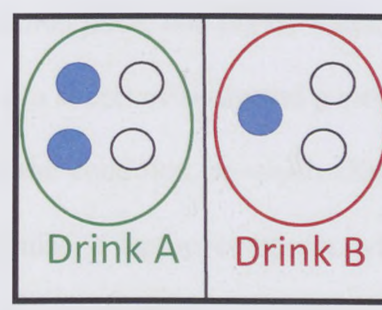
Will one drink taste sweeter? (Direct relation)

Will one drink taste more watery? (Inverse relation)

(b) Water varied/sugar controlled

Computer diagrams

Manipulatives



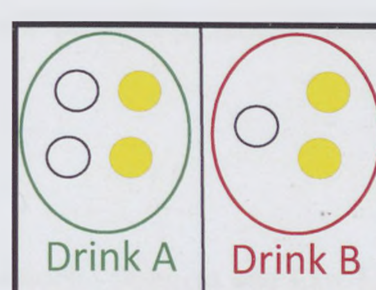
Will one drink taste more watery? (Direct relation)

Will one drink taste sweeter? (Inverse relation)

(c) Lemon controlled/sugar varied

Computer diagrams

Manipulatives



Will one drink taste sweeter? (Direct relation)

Will one drink taste more sour/sharp? (Inverse relation)

(d) Lemon varied/sugar controlled

Computer diagrams | Manipulatives

Will one drink taste more sour/sharp? (Direct relation)

Will one drink taste sweeter? (Inverse relation)

Figure 3.1 (a–d) Items used in computer diagrams and manipulatives conditions

3.2.3 Materials

Each condition required a different set of materials. In the physical demonstration condition the following materials were used: one jug of lemon juice; one jug of water; two large clear plastic glasses; one bowl of sugar cubes; and two sheets of laminated paper, the first labelled ‘Drink A’ and the second ‘Drink B’. For the condition in which children viewed computer diagrams depicting the mixing of drinks, a laptop computer with a PowerPoint™ presentation displayed the items. In the manipulatives condition, the following materials were used: one red box and one green box; two sheets of laminated paper, the first labelled ‘Drink A’ and the second ‘Drink B’; and different coloured counters to represent different ingredients (white, blue and yellow).

In each condition, the researcher used the same interview protocol to record children’s responses and justifications (See Appendix 3.1).

3.2.4 Ethical Approval

The design of the study gained ethical approval from the Social Sciences and Law Ethics Committee at Oxford Brookes University. The procedure for gaining access to schools was as follows. An initial letter was sent to the head teachers of all the schools involved (Appendix 3.2) containing a suggested letter of consent to be sent out to the parents of the children (Appendix 3.3). The head teacher of each school acted as the gatekeeper and was free to decide whether or not to seek permission from parents for their children's participation in the study.

3.2.5 Procedure

In a whole class briefing, the class teacher introduced the researcher to the children. Each child then worked with the researcher on a one-to-one basis at a table outside the classroom. Before the task began each child was told that they would play a game about what drinks would taste like if they were mixed in different ways. The child was encouraged to talk about the types of drink they liked at home, and the researcher would use this as an example of what was going to be done. For example, if the child said they liked to drink tea at home, the researcher would say that to make a drink of tea taste nice you need to use water, tea bags and sugar etc. It was then explained that they would thinking about the type of drink where different things were mixed together.

The researcher gave the child the first problem and explained, 'We are going to pretend to mix some water and sugar together.' The researcher proceeded to explain the question, i.e. whether the water/lemon was the same amount for drink A and B and how much sugar (number of sugar cubes) would be added to each drink. Each child explained their judgements. After the child completed all the items, they returned to the classroom.

3.4 RESULTS

The results of Experiment 1 were analysed by age using a mixed measures ANOVA with age and mode of presentation as between subjects' factors. The reason for taking this approach is that within the age ranges tested (4–9 years) the use of manipulatives may have different effects. Manipulatives may only facilitate problem solving when the children already have some understanding of the relation (direct or inverse) in question. On this basis manipulatives may aid younger children with direct relations problems, while having little impact on their performance with inverse relations problems. For older children already competent with direct relations, the impact of manipulatives may be found in their performance on inverse relations problems only.

3.4.1 Analysis of Main Effects

Table 3.3 shows the general descriptive statistics for each age group by condition with mean scores displayed for both direct and inverse items.

Table 3.3 Mean Scores for Each Problem Type by Age and Condition

Condition	Age (N)	Type of relation	
		Direct (Max 4) Mean *(SD)*	Inverse (Max 4) Mean *(SD)*
Physical demonstration	4 years (10)	0.90 *(0.88)*	1.50 *(1.08)*
	5 years (17)	1.76 *(1.52)*	1.12 *(0.93)*
	6 years (10)	1.00 *(1.55)*	0.70 *(0.95)*
	7 years (10)	2.30 *(1.06)*	1.50 *(1.35)*
	8 years (20)	2.60 *(1.10)*	2.65 *(1.23)*
	9 years (12)	2.83 *(1.36)*	2.75 *(1.14)*
Computer diagrams	4 years (9)	1.44 *(1.01)*	0.89 *(0.78)*
	5 years (16)	1.88 *(1.26)*	1.25 *(1.18)*
	6 years (10)	1.60 *(1.17)*	1.60 *(0.70)*
	7 years (10)	2.80 *(0.92)*	1.50 *(1.27)*
	8 years (22)	3.27 *(0.77)*	2.50 *(1.30)*
	9 years (11)	2.91 *(1.04)*	2.73 *(1.19)*
Manipulatives	4 years (9)	0.89 *(1.05)*	0.44 *(1.01)*
	5 years (12)	2.33 *(1.23)*	0.92 *(0.10)*
	6 years (10)	2.70 *(1.06)*	2.10 *(1.52)*
	7 years (10)	3.60 *(0.97)*	2.30 *(1.64)*
	8 years (21)	3.38 *(0.86)*	2.86 *(1.24)*
	9 years (9)	3.44 *(0.73)*	3.33 *(0.87)*

The analysis showed that two distributions were significantly skewed. For 7 year old children receiving problems with manipulatives, direct relations problems were significantly negatively skewed (z=-2.66) suggesting this group found the direct relations problems very easy. For 4 year olds receiving problems with manipulatives, there was significant positive skew (z=2.51) on inverse relations problems suggesting these problems were very difficult for them. It is recommended by Howell (1997) that data transformation can help to reduce the skew of distributions. As both positive and negative skew were found in the current data set an Arcsine transformation was carried out. The Arcsine transformation improved the negative skew observed in the 7 year olds to borderline significance (z=-2.01) while the positive skew of 4 year olds remained significant (z=2.68). Howell (1997) reports though that ANOVA is robust enough to cope with some deviation from the normal distribution.

Mauchly's diagnostic test of sphericity was shown to be non-significant which means that unadjusted F ratios can be reported. Levene's test of equal variances showed that equal variances could not be assumed as significant differences were observed for direct and inverse relations ($p<.05$). This means that it is important to choose post-hoc tests which can cope with this deviation. Field (2000) recommends that when equal variance cannot be assumed and when sample sizes differ, Games-Howell post-hoc tests provide the most robust measures.

There was a significant main effect of type of relation ($F_{1,210}= 35.23, p<.0001$). Games-Howell post-hoc tests showed that direct items were generally easier than inverse items ($p<.0001$) and a Cohen's d effect size of 0.26 SD, which is considered a small effect size (Cohen, 1988). There was also a significant interaction of relation by mode of presentation ($F_{2,210}= 3.58, p<.030$). Figure 3.2 below shows the relation by mode of presentation interaction. It shows that children performed better in direct relations problems in both the computer diagrams and manipulatives condition, but there is little difference in performance on direct and inverse relations items in the physical demonstration condition.

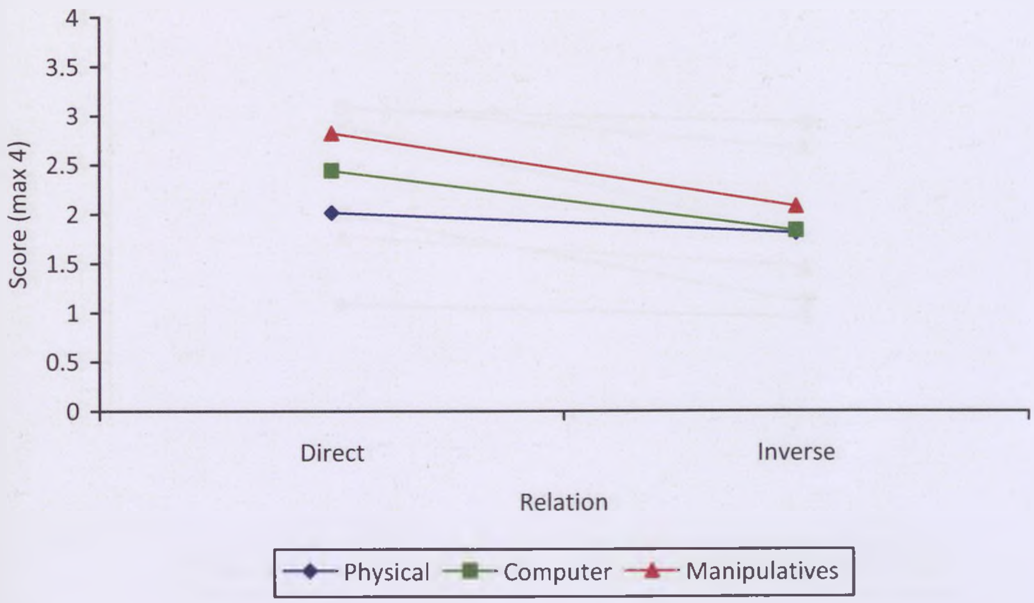


Figure 3.2 Interaction of mean scores on direct and inverse relations problems by condition

There was also a significant age by relation interaction ($F_{5,210}= 4.44$, $p<.001$). Figure 3.3 shows the age by relation interaction. As can be seen, the explanation for the interaction appears to be that the children aged 5, 7, and 8 years performed better on direct than inverse relations problems. At ages 4, 6, and 9 years the children showed comparable levels of performance on direct and inverse relations problems. These differences show that the significance of the distinction between direct and inverse relations is age-dependent. The inclusion of children for which these problems were too difficult (4 years) or too easy (9 years) probably explains the small effect size observed for the significant effect of relation.

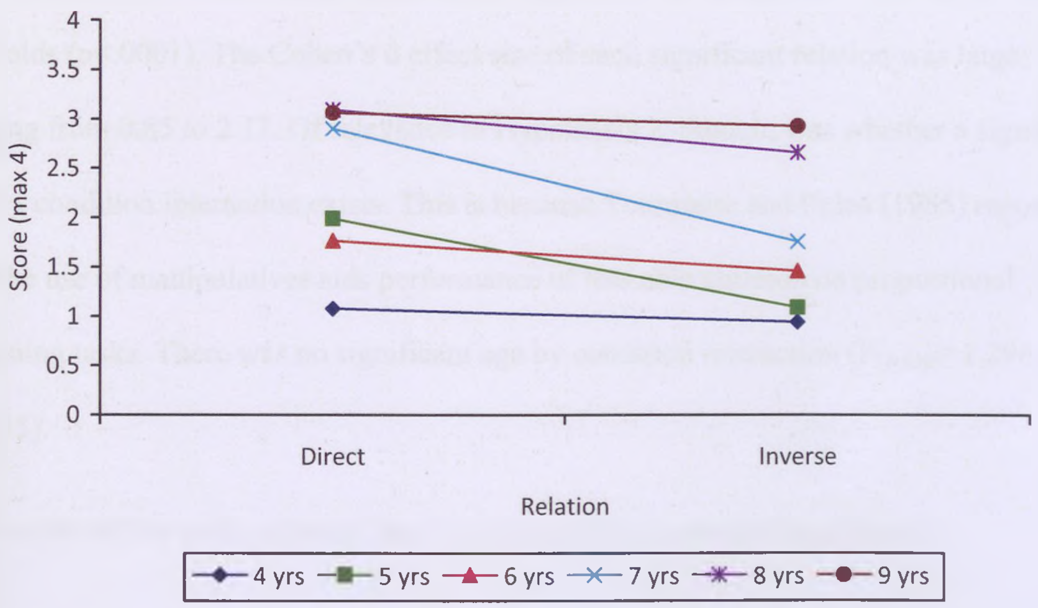


Figure 3.3 Interaction of mean scores on direct and inverse relations problems by age

3.4.2 Between Subjects Factors

On the tests of between subjects effects there was a significant between subjects effect of form of presentation ($F_{2,210}$= 7.19, p<.001). Games-Howells post-hoc tests show that children in the manipulatives condition scored significantly higher than children receiving physical demonstrations (p<.006). The Cohen's d effect size for this was 0.46, which is considered a medium effect size. It must also be noted that children shown problems in the computer diagrams format did not differ significantly from children in either the physical demonstration or manipulatives conditions. These results confirm that it is not possible to accept Hypothesis 1:

'Significant differences will be observed between children receiving physical demonstrations and those receiving computer diagrams on direct and inverse relations problems'

There was also a significant between subjects effect of age ($F_{5,210}$= 23.027, p<.0001). This is expected considering the broad range of ages tested. Games-Howells post-hoc tests revealed that children aged 7 years scored significantly higher than 4 year olds (p<.0001) and 5 year olds ($p<.016$). Children aged 8 and 9 years scored significantly higher than 4–6 year olds ($p<.0001$). The Cohen's d effect size of each significant relation was large, ranging from 0.85 to 2.17. Of relevance to Hypothesis 2, though, was whether a significant age by condition interaction exists. This is because Tourniaire and Pulos (1985) reported that the use of manipulatives aids performance of less able children on proportional reasoning tasks. There was no significant age by condition interaction ($F_{10,210}= 1.296$, $p<.235$).

The results of this analysis mean that it is not possible to accept Hypothesis 2:

'Children receiving problems with the aid of manipulatives will perform significantly better than those children who do not on direct and inverse relations problems'

Although direct relations problems presented with manipulatives were significantly easier than those presented with physical demonstrations, the use of manipulatives did not provide any specific advantage over the use of computer diagrams.

3.4.3 Summary of Results from Experiment 1

Experiment 1 had two aims. The first was to test whether presenting relational thinking problems as computer diagrams would provide data comparable to the results that would be expected given children's day-to-day experience with intensive quantities. It was shown that the children receiving problems in the form of physical demonstrations did not score significantly differently to those children shown computer diagrams (rejecting Hypothesis 1). This suggests that the use of computer diagrams for the current research provides an ecologically valid and practical means by which to assess children's performance on intensive quantity problems.

The second aim of Experiment 1 was to test whether the use of manipulatives in problem presentation could increase children's performance on direct and inverse relations problems. Tourniaire and Pulos (1985) reported a general increase in the performance of low ability children on proportional reasoning problems when manipulatives were used. In the current study, the manipulatives condition provided a limited advantage over the physical demonstration condition for direct relations problems. As the manipulatives condition did not appear to offer any advantages on inverse relations problems, it is possible that the improved performance resulted not from a new logical insight but to a comparison of manipulatives numerically, which would always lead to the correct answer on direct relations problems. It must also be noted that this advantage on direct relations problems did not hold when compared to children shown computer diagrams. Due to a lack of general improvement in performance when manipulatives were used, it was not possible to accept Hypothesis 2.

3.5 Experiment 2: Development of the Logical Relations of Intensive Quantities

The aim of the second experiment is to revisit a study initially reported by Strauss and Stavy (1982), which described the developmental progression of four relations that children need to build a successful concept of intensive quantities. Strauss and Stavy identified and compared: direct relations, inverse relations, sampling, and proportional equivalence.

Strauss and Stavy (1982) tested the development of these four types of relations between quantities within the setting of sweetness using comparisons of sugar/water solutions. They reported a developmental progression from direct relations through inverse relations and sampling to proportional equivalence. Strauss and Stavy also reported that increasing success with inverse relations and proportional equivalence problems begins at roughly the same age (9 years). Their analysis is important as they identified four essential logical parameters needed to understand intensive quantities. It must be noted, though, that Strauss and Stavy's aim was not to test the development the understanding of intensive quantities in general but specifically children's understanding of sweetness. To test whether their findings can be extended to the wider concept of intensive quantity it is necessary to study the broader concept of taste. The main difficulty that arises when trying to relate Strauss and Stavy's findings to the wider context of intensive quantity understanding is that direct relations problems about sweetness always involve comparisons between quantities of sugar while inverse relations problems about sweetness always involve comparisons between volumes of water. To extend Strauss and Stavy's work to the development of the general intensive quantity concept of taste it is essential to vary systematically the type of relation and the quantity.

Desli (1999) addressed this issue in part. She compared children's performance on direct and inverse relations while controlling for language. Experiment 2 extends Desli's analysis by comparing the development of the intensive quantity concept across the four types of logical relation identified by Strauss and Stavy (1982). This analysis is necessary to test whether the link that Strauss and Stavy established between the development of inverse relations and proportional relations holds when language is controlled.

Before moving on to the hypothesis it must be noted that Experiment 2 does not include data on children who receive items using manipulatives. As the data from physical demonstrations and computer diagrams did not differ significantly during Experiment 1 the scores from these groups are combined for the analysis of Experiment 2.

3.5.1 Hypothesis

The main finding from Strauss and Stavy's (1982) study is used to generate the experimental hypothesis.

Hypothesis:

'In taste settings, the order of relative difficulty of different relations will follow Strauss and Stavy's (1982) observations in the context of sweetness from direct relations, inverse relations, sampling, and proportional equivalence'

3.6 Methodology

3.6.1 Participants

A sample of 141 children (67 boys and 74 girls) from eight Oxfordshire nursery and primary schools participated in Experiment 2. The sample consisted of six different age groups displayed below in Table 3.4.

Table 3.4 Participant Summary

Age group	No. of children	Age range (years:months)	Mean age *(SD)*
4	17	4:00–4:11	4:06 *(.04)*
5	27	5:03–5:11	5:08 *(.02)*
6	15	6:00–6:11	6:07 *(.05)*
7	17	7:00–7:11	7:06 *(.04)*
8	42	8:00–8:11	8:06 *(.04)*
9	23	9:00–9:09	9:05 *(.02)*

3.6.2 Design

Experiment 2 employs a mixed model 6×4 design: Age (4 years vs. 5 years vs. 6 years vs. 7 years vs. 8 years vs. 9 years – between participants) by relation (direct relations vs. inverse relations vs. sampling vs. proportional equivalence – within participants). A total of twelve items incorporating two taste situations (sugar/water and sugar/lemon) comprised Experiment 2. Four of the items measured performance on direct relations when different quantities were controlled. Four items measured performance on inverse relations when different quantities were controlled. The direct and inverse relations items were used in Experiment 1 (see Table 3.2). The analysis of Experiment 2 included four extra items. Two items measured children's performance with sampling and two items measured performance on proportional equivalence (see Figure 3.4 a–d).

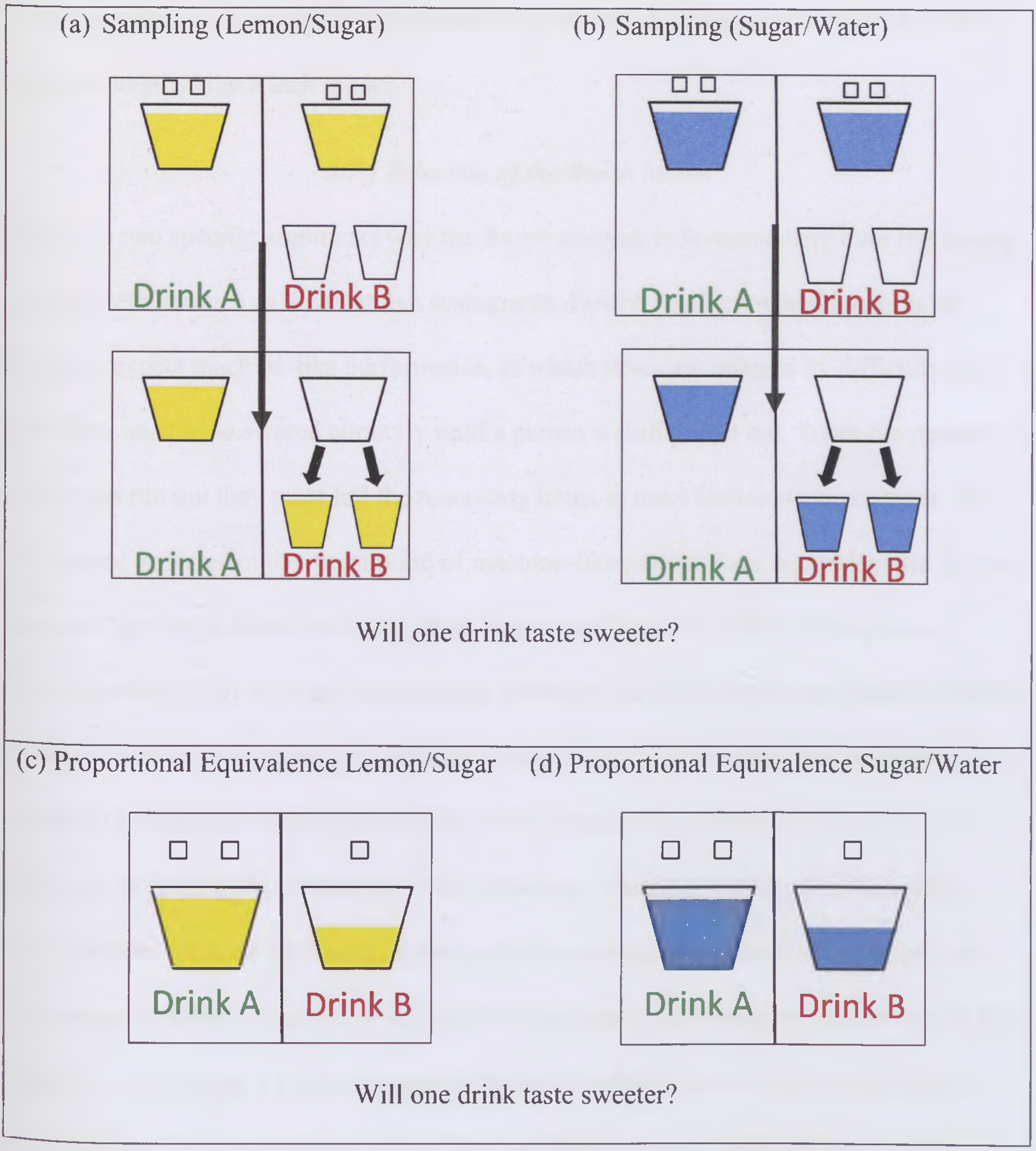


Figure 3.4 (a–d) Examples of the additional Sampling and Proportional Equivalence items

3.7 Results

The aim of Experiment 2 was to examine whether the order of relations identified by Strauss and Stavy (1982) in relation to sweetness can be extended to the intensive quantity of taste. The analysis of Experiment 2 tests whether the different logical ideas that Strauss and Stavy identified will scale along a single 'intensive quantity' ability. Also of interest is the amount of progress children with high enough ability to solve inverse relations problems make with proportional equivalence problems. To address these aims the data analysis employs the Rasch model.

3.7.1 Selection of the Rasch Model

There are two specific arguments why the Rasch analysis is favoured here over traditional scaling methods such as the Guttman scalograms. Firstly, scaling methods such as the Guttman expect machine-like performance, in which items are ordered by difficulty and each item must be answered correctly until a person's ability runs out. When the person's ability has run out they must fail the remaining items to meet Guttman expectations. For the current experiment the assumption of machine-like performance is problematic for two reasons. The first is based on reports from Strauss and Stavy's (1982) original data. Although they report an order of increasing difficulty from direct relations through inverse relations and sampling to proportional equivalence, they also reported that younger children (4 years) can make correct judgements on sampling problems. If these observations were to be replicated in this experiment, Guttman scaling would exclude these children whereas the Rasch model can accommodate this type of response pattern. The second problem is that due to the nature of the items, machine-like performance is not expected. Experiment 2 comprises groups of items, and the focus of interest is whether a child's ability 'runs out' when engaged on a particular group of items rather than an individual item. For example, Experiment 2 tests whether 9 year olds are able to solve direct and inverse items but fail sampling and proportional equivalence items; it is of less importance for the aims of the experiment whether a 9 year old makes a mistake on one direct relations item, perhaps due to loss of concentration. The Rasch analysis can accommodate these types of errors more easily than the Guttman scaling, which has stricter parameters of acceptable performance. The second related argument for using Rasch analysis rather than Guttman scalograms stems from the fact that some items are expected to show similar levels of difficulty. With Guttman scales, two items of equal difficulty cannot be included within the same scale. As the current experiment includes groups of items expected to have the same level of difficulty (i.e. direct relations), a scaling model that can cope with this reality is required.

3.7.1.1 Important Principles of Rasch Analysis

Before reporting the results of Experiment 2, it is important to give a brief overview of the ideas important to Rasch analysis as it will help with the interpretation of the subsequent sections.

The Rasch model is based on the idea that the items in a particular test are trying to measure a specific underlying ability (unidimensionality). In this case it is the children's understanding of the intensive quantity concept. The Rasch model works with total scores of items and participants to estimate the probability of passing items. As Bond and Fox (2001) point out:

"The model is based on the simple idea that all persons are more likely to answer easy items correctly than difficult items, and all items are more likely to be passed by persons of high ability than by those of low ability." (p.xix)

The Rasch model consists of two parts: the first is the calibration of item difficulties and person ability; the second is the estimation of fit.

3.7.1.2 Test Calibration

The calibration of the test involves ordering items and participants by success rates. Next the model removes any items and participants with ceiling- or floor-level performance. This is because such items and participants reveal little about the test. For example, if there are items which are too easy for the majority of the children, then the recommendation is to collect more data with less able participants. In the case of the current experiment this would mean asking younger children. The data on the remaining items and participants are then converted to the same log odds scale. The benefit of this over reporting percentages is that the distances between the items and the children are more meaningful. Knowing, for example, that 7 year olds score higher than 6 year olds is interesting, but knowing how much more ability is needed to reach a comparable level is even more interesting.

Bond and Fox (2001) point out that:

"The logit scale is an interval scale in which the unit intervals between the locations on the person-item maps have a consistent value or meaning. The Rasch model routinely sets at 50% the probability of success for any person on an item located at the same level on the item-person logit scale." (p.29)

3.7.1.3 Estimation of Model Fit

After the data has been ordered along the logit scale the Rasch model calculates how well the observed data 'fits' onto this model. As Bond and Fox (2001) explain:

"Rasch analysis provides indicators of how well each item fits within the underlying construct. This is a crucial aid for the investigator assessing the meaning of the unidimensional construct. That is, fit indices help the investigator to ascertain whether the assumption of unidimensionality holds up empirically. Items that do not fit the unidimensional construct (the ideal straight line) are those that diverge unacceptably from the expected ability/difficulty pattern. Therefore, fit statistics help to determine whether the item estimations may be held as meaningful quantitative summaries of the observations (i.e., whether each item contributes to the measurement of only one construct)." (p.26)

The calculation of fit is based on residual scores, the question being how much a person/item over- or underperforms from what is known about their ability/difficulty. Two measures (infit and outfit) provide an estimation of participant and item fit. The researcher decides which is the most appropriate and meaningful statistic to report. The major difference between infit and outfit is that outfit is based on the chi-square statistic and is sensitive to the presence of outliers, while infit is a weighted score which reduces the impact of outliers. Both measures are reported as mean squares, with the stated values indicating whether a particular person or item shows too much or too little variation from the Rasch model's predictions.

As Bond and Fox (2001) explain:

"*An infit or outfit mean square of 1+x indicates 100x% more variation between the observed and the model predicted response patterns than would be expected if the data and the model were perfectly compatible. Thus, an infit mean square value more than 1, say 1.30 (1+.30) indicates 30% more variance in the observed data than the Rasch model predicted. An outfit MSQ value less than 1 say, 0.78 (1-.22=.78) indicates 22% less variation in the observed response pattern than was modelled.*" (p.177)

3.7.2 The Results

The results will be reported in two sections. The first section presents data from the Rasch analysis concerning the items' compatibility and subsequent scaling of items by difficulty. The relation between item difficulty and age-related progress is also identified.

The second section will build on the results reported in the first section. The results in the second section are qualitative in their nature. They consider the justifications that children of different ages gave to each type of item (direct, inverse, sampling and proportional). The analysis will be used to explain the progress and difficulties reported by the Rasch analysis at different ages. The analyses of justifications provide an exploratory analysis of whether Strauss and Stavy's (1982) observation that solving inverse relations problems leads to progress on proportional equivalence problems can be replicated in an intensive quantity setting.

3.7.3 Section 1: Rasch Analysis

The results of the Rasch analysis are presented in two stages. The first stage looks at the item data. This will allow conclusions to be made about the compatibility of the items. The question here is whether the items measure the same underlying concept. The Rasch model's ordering of items will provide a test of the first experimental hypothesis: whether the ordering of items that Strauss and Stavy (1982) originally observed in relation to sweetness extends to the intensive quantity of taste.

The second stage of the analysis reports the participant data and will establish the link between age and performance on the different logical ideas of intensive quantities. This analysis will establish whether the ages of understanding reported by Strauss and Stavy (1982) in relation to sweetness can be extended to taste.

3.7.3.1 Selection of Rasch Model and of Appropriate Fit Parameters

A partial credit Rasch model was chosen to analyse the data; for each item presented a possible score of two was available. This allowed a greater level of discrimination between items and more accurately reflected shifts in ability necessary to solve problems of different logical relations. For each of the 12 items the children received one mark for a correct judgement and an additional mark for a correct justification. It was considered important to incorporate data on justifications to help discriminate between partial success and a full understanding because, as Strauss and Stavy's study reported, even young children can make correct judgements on sampling problems.

As children's scores were based on multiple choice and interview data, the section of the appropriate fit parameters were set to 0.6–1.4 mean squares, as recommended by Wright and Linacre (1994, cited in Bond and Fox, 2001), for survey data. For the subsequent statistical analysis of the data the BigSteps™ Rasch analysis program was used (Linacre and Wright, 1991).

3.7.3.2 Item Fit Statistics

The results of the Rasch analysis are reported using the infit statistics. The primary reason is based on Strauss and Stavy's investigation, in which young children make correct judgements about sampling items. Outliers are expected and the infit statistic is designed to reduce the impact of expected outliers (Linacre and Wright, 1991). The data from the partial credit model with regard to the item data is shown in Table 3.5.

Table 3.5 Item Diagnostics Showing Infit Statistics

	Raw Score	Count	Measure	Model Error	Infit	
					MNSQ	ZSTD
Mean	117.6	135	0.00	.12	.97	-0.3
SD	46.2	0.00	0.59	.01	.17	1.5
Max	191	135	1.10	.15	1.35	2.9
Min	40	135	-.90	.10	.74	-2.0
Item reliability .96						

The maximum and minimum infit scores show that the items are within the acceptable ranges (0.6–1.4) of the Rasch model, meaning that the data is well matched to the model's expectations. A high item reliability index of .96 means that if these items were given to another group of children, there is a very high likelihood that the same order of item difficulty would be observed.

3.7.3.3 Item Difficulty

Figure 3.5 shows the order of difficulty of the 12 test items. The general order of item difficulty has direct relational problems as the easiest to solve, then inverse relational, then sampling, and finally proportional equivalence problems representing the most difficult problem type.

It must be noted that despite the general pattern of direct items being easier than inverse items, for two out of eight items the opposite relation was observed. When the children needed to infer direct relations from the relation between water and taste, the items became relatively difficult. When presenting the inverse relations problems involving the relation between water and taste, the item became relatively easy. The section on children's justifications examines the reasons behind these anomalies.

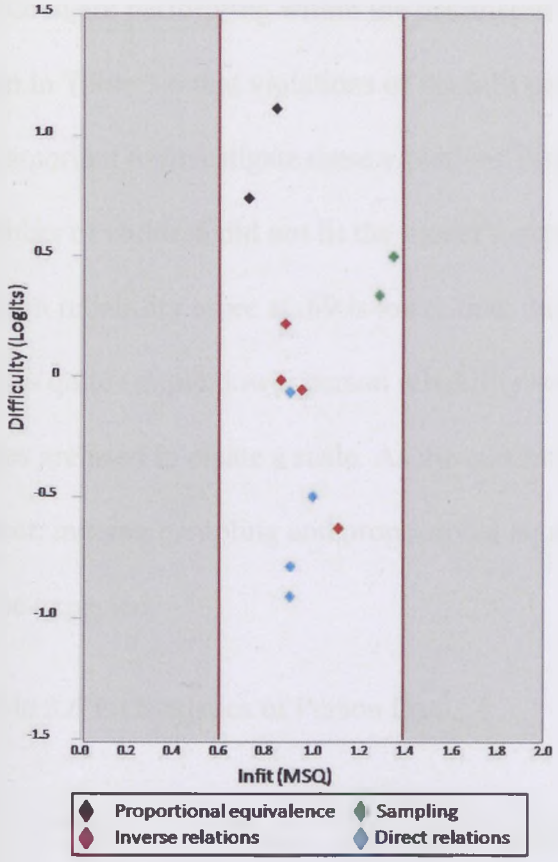


Figure 3.5 Relative difficulties of the items

3.7.4 Person Measures

The children's data is presented in two parts. The first assesses the general fit of the participants to the model. This will allow a judgement to be made on the suitability of the model in explaining the development of children's understanding of the logical relations of intensive quantities. This stage will also include the identification and examination of any participant data not deemed compatible with the model.

The second part of the data analysis maps the age-related ability levels onto the item difficulty scale, giving a clearer idea of the progress made with the intensive quantity concept at different ages.

3.7.4.1 Participant Fit Statistics

The fit statistics for the children's data are displayed below in Table 3.6. Of particular interest are the maximum and minimum infit scores, which indicate whether all the children are performing within the parameters of the Rasch model's expectations. It can be seen in Table 3.6 that violations of the infit parameters (0.6–1.4 mean squares) occurred. It is important to investigate these violations further to assess whether an unacceptable number of children did not fit the model's expectations. It must also be noted that the person reliability score at .69 is lower than the item reliability score at .96. The reason for this is quite simple: lower person reliability scores are said to occur when only a few item types are used to create a scale. As the current study consisted of four types of items – direct, inverse, sampling and proportional equivalence – a low level of person reliability is to be expected.

Table 3.6 Fit Statistics of Person Data

	Raw Score	Count	Measure	Model Error	Infit	
					MNSQ	ZSTD
Mean	10.4	12	-0.23	.39	0.98	-0.2
SD	5	0.00	0.72	.06	0.29	.8
Max	22	12	1.73	.63	**1.94**	2.7
Min	2	12	-1.67	.35	**0.53**	-2.0
Person reliability 0.69 excluding 6 extreme scores (3 children achieved maximum scores and 3 children received minimum scores).						

3.7.4.2 Analysis of Children Not Compatible with the Rasch Model

A total of 15 children (10.6% of the total sample) were identified as not fitting the expectations of the Rasch model. The data on these children is displayed in Table 3.7. The data has been ordered by children's infit mean squares score. As can be seen, of the 15 children with misfitting data, 11 are reported as having underfit. This means the performance of these children displayed more variation than is acceptable. Only four children were reported as displaying overfit, indicating less variation than expected. The important point in assessing the severity of the misfitting data is to look for evidence that several misfitting children produce similar misfit response strings. This evidence could then indicate a developmental pathway different from that of the Rasch model's predictions and would require further investigation.

The only pattern to emerge from the misfitting data of the current experiment occurred among a very small sample of the underfitting children (N=4). These children solved difficult proportional equivalence problems while failing easier sampling problems. This would suggest that although these children had enough ability to solve sampling problems, they lacked the specialist knowledge required. As this pattern of response only occurred in a few children, it does not require further investigation.

Table 3.7 Children with Fit Score Beyond the Acceptable Thresholds

	Child id (age category)	Infit score (MSQ)	Item (Score: 0,1,2)			
			Direct	**Inverse**	**Sampling**	**Equivalence**
Underfit	714 (9)	1.91	0 0 2 0	2 0 2 2	0 0	2 0
	336 (7)	1.79	2 2 0 2	2 2 2 0	2 2	2 2
	322 (8)	1.78	2 2 2 0	2 2 2 2	2 2	2 2
	342 (8)	1.78	2 2 2 0	2 2 2 2	2 2	2 2
	638 (5)	1.74	0 0 2 0	0 0 0 0	0 0	0 2
	333 (8)	1.71	0 2 0 2	2 2 2 2	0 1	2 2
	738 (5)	1.68	0 0 2 2	0 0 2 2	0 0	0 0
	704 (8)	1.65	0 2 2 2	2 2 2 2	2 0	2 2
	706 (9)	1.63	2 0 2 2	2 2 2 2	0 0	2 2
	643 (9)	1.52	2 2 0 1	0 2 2 1	0 0	2 2
	749 (6)	1.50	0 0 0 0	0 0 2 2	1 1	0 0
Overfit	748 (6)	0.59	2 2 0 0	2 0 0 0	1 1	0 0
	404 (4)	0.59	2 0 0 0	1 0 0 1	1 0	0 0
	113 (6)	0.57	0 0 0 0	1 0 1 1	1 1	0 0
	338 (8)	0.53	2 2 0 2	2 0 1 0	0 0	0 0

3.7.4.3 Age-Related Progress

As it is reasonable to assume that the Rasch model provides an accurate guideline as to the relative difficulty of the items and of the ability of the children at different ages, it is important to look at how children's performance on average changes with age. Figure 3.6 presents the original items in order of difficulty with lines added to represent the mean ability scores for each age group.

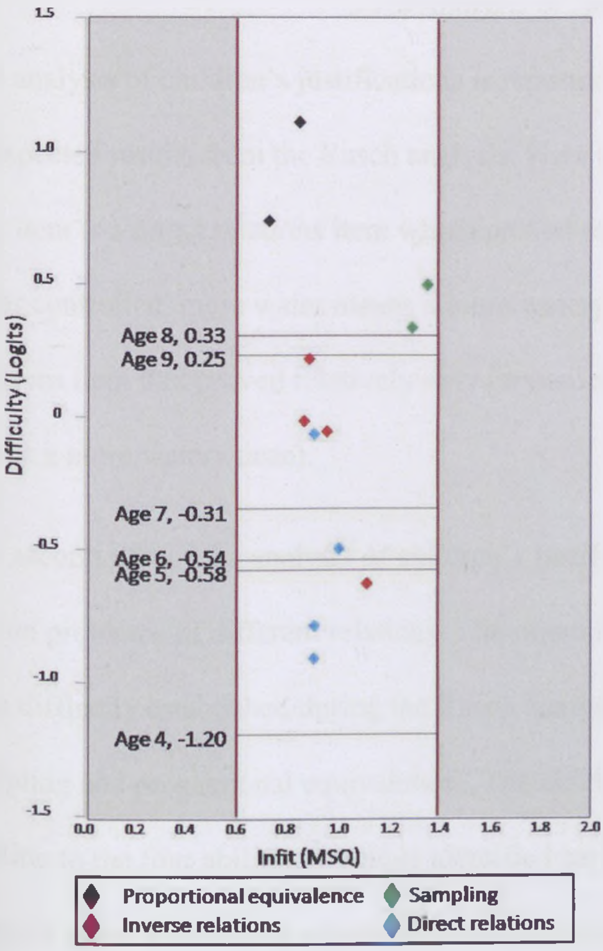


Figure 3.6 Item difficulties with mean age ability scores

The results from this sample seem to indicate three main developments in ability tied to age. These appear between 4 and 5 years, between 6 and 7 years and finally between 7 and 8 years. The placement of the mean ability scores indicates that at age 4, the majority of children have little understanding of the task and are unable to reason consistently on even the simplest items (direct relations). At around 5 years, the children begin to work with direct relations, but they do not fully understand them before the age of 7 years. The results of children of 8 and 9 show that once direct relations are mastered, they can progress quite quickly to an understanding of inverse relations. However, two variables – sampling and proportional equivalence reasoning questions – remain difficult even for these older children; the second part of the results provides a more detailed examination of these age/performance differences. This section will use the children's judgements to explain changes in ability with age.

3.7.5 Analysis of Justifications

The analysis of children's justifications is reported in two parts. The first part considers unexpected results from the Rasch analysis. Here two items receive greater attention. The first item is a direct relations item which proved relatively difficult (water levels varied, sugar controlled, more water means a more watery taste). The second item is an inverse relations item that proved relatively easy (water levels controlled, sugar varied, less sugar means a more watery taste).

The second part of the analysis of children's justifications considers their justifications by age on problems of different relations. The organisation of this part is by the order of the item difficulty established during the Rasch analysis (direct relations, inverse relations, sampling and proportional equivalence). The children's justifications are displayed in relation to the four ability groupings identified earlier. The first, or least able, group consists of the 4 year olds with a mean ability score of -1.2 logits. The second ability group consists of the 5 and 6 year olds with mean ability scores of -.58 and -.54. Next are the 7 year olds whose mean ability score is -.31 logits. The final ability group consists of the 8 and 9 year olds with mean ability scores of .33 and .25 respectively.

3.7.5.1 Definition of Codes

A total of nine different justification codes were used to categorise the children's justifications; the list of codes is shown in Table 3.8 (justifications marked in *italics* were only possible for sampling questions).

Table 3.8 Different Responses Used by Children to Justify their Judgements

Label	Examples of responses
Unclear	'I don't know', 'I'm not sure', 'Because it tastes nice' or refusal to give a justification
Same Drink	'They are both just sugar and water (and so will have the same taste)'
Controlled variable	'Both drinks will be as sweet as each other because both have one sugar'
Direct	'Drink A is sweeter because it has more sugar' 'Drink A is sweeter because it has more water'
Inverse	'More water will make it less sweet' 'less sugar will make the lemon more sweet'
Proportion	'One sugar with 1/2 glass of water will have the same taste as a full glass with 2 sugars', 'It has less water and less sugar so it will taste the same', 'It is the same just half the amount'.
Original state	*'You just poured them', 'It is still the same (drink) as before (you poured it)'*
No sugar	*'This drink has got no sugar in', 'The sugar is in the bottom of this one (the drink poured second)'*
Same sugar	*'They all have 2 sugars in them' (Sampling justifications using absolute rather than relative judgements can lead to a correct or incorrect judgement).*

3.7.5.2 Analysis of Unexpected Results

There were two items which scaled differently on the Rasch model to expectations, given the difficulty of items requiring reasoning about similar relations. Both items involved the consideration of taste, and included the language 'more watery'. In this context children appeared to underperform on the direct form of this problem (water levels varied, sugar controlled, more water means a more watery taste), while children over-performed on the inverse form of this problem (water levels controlled, sugar varied, less sugar means a more watery taste). Table 3.9 shows the children's judgements and justifications to these items.

Table 3.9 Children's Justifications on Direct and Inverse Problems Using the Expression 'More Watery'

Judgement	n (proportion of responses	Direct Relations item (Proportion of justification by judgement)				
		Unclear	Direct	Inverse	Controlled variable	Same drink
Same	65 (.46)	.05	.01	.01	.35	.04
More A (correct)	59 (.42)	.02	.38	.01	-	-
More B	17 (.12)	.04	.01	.07	-	-
		Inverse Relations item (Proportion of justification by judgement)				
		Unclear	Direct	Inverse	Controlled variable	Same drink
Same	13 (.09)	.04	-	-	.04	.01
More A	32 (.23)	.08	.13	.02	-	-
More B (correct)	95 (.68)	.10	.10	.48	-	-

Looking first at the justifications given on the direct relations problem, the majority of the answers stem from children reasoning that both drinks taste the same (N=65). The majority of children justified this response with the argument that the same amount of sugar means the same taste (.35). This gives a strong indication that quantity salience played a key role in increasing the difficulty of this problem, as a large proportion of the children considered sugar as the only taste variant.

Looking at the justifications for the inverse form of this problem (which children found relatively easy), the majority of correct responses included the inverse justification of less sugar leading to a more watery taste (.48). As it is clear from the children's justifications to the direct form of this problem that they did not consider the role of water in the problem, it is possible this same variable salience reduced the difficulty of the inverse relations problem. If the children see sugar as the only factor influencing taste, then it is possible to reason about the inverse problem in terms of direct relations: less sugar means less taste.

This interpretation becomes more probable as the same patterns of difficulty were not shown for items regarding the taste of lemon/sugar solutions. With lemon and sugar problems children had to consider the role of two competing strong tastes. It would suggest that salience can play a role in children's thinking, but that it is context dependent.

3.7.5.3 Justifications in Relation to Progress Along the Rasch Model

It is important now to look at the justifications offered at different ability levels in relation to different groups of problems. Although the data in this section is reported by age, the justifications are discussed in terms of the four age-related ability levels reported by the Rasch model (4 year olds, 5 and 6 year olds, 7 year olds, 8 and 9 year olds).

3.7.5.4 Children's Justifications by Age to Direct Relations Problems

Table 3.10 displays the proportion of justifications used in response to judgements made about items of direct relations by children at different ages. The type of reasoning that could (but did not necessarily) lead to the correct answer is displayed in bold. The low ability group had a mean logit score lower than that of the easiest direct relations item, indicating that reasoning about direct relations in taste settings is too difficult for this group of children. This prediction is substantiated by the fact that just under a third of the children (.31) cannot provide an explanation, while just under a quarter of 4 year olds (.24) are able to reason explicitly about direct relations. The majority of justifications at this ability level are based on highlighting the similarities between two drinks as a basis for explaining the taste.

The justifications of the next ability group (5 and 6 year olds) show that this increase in ability is due to the use of more accurate justifications, with almost half of justifications at this level involving direct reasoning (.48 and .43 for 5 and 6 year olds respectively). As with the 4 year olds, the most common misconception was to reason on the basis of similarities (e.g. 'they both have the same amount of lemon juice so they will both have the same taste'). For the third ability grouping (7 year olds) their justifications for direct relations problems are an indication of the progress made at this age. The justifications of 7 year olds include a higher proportion of explicit references to direct relations (.66). There is little to distinguish this ability grouping from the highest ability grouping (8 and 9 year olds) on direct relations items. The only difference is a steady decrease in the proportion of respondents focusing on the similarities of drinks in the highest grouping. This lack of distinction between these two highest ability groupings on direct relations problems is perhaps not surprising, as the results based on the Rasch model predicted that in general the top two groups possessed enough ability to pass direct relations problems.

Table 3.10 Proportion of Justifications Used by Children at Different Ages to Solve Problems of Direct Relations

Ability group	Age	Unclear	Same drink	Controlled variable	Direct	Inverse	Proportion
1	4	.31	.06	.28	**.24**	.12	-
2	5	.11	.02	.30	**.48**	.10	-
	6	.07	-	.37	**.43**	.13	-
3	7	.06	.02	.28	**.66**	.04	-
4	8	.05	.01	.12	**.70**	.11	.01
	9	.04	-	.10	**.66**	.17	.02

3.7.5.5 Children's Justifications by Age to Inverse Relations Problems

Table 3.11 summarises what proportion of the different age groups offered each type of justification about their judgements to inverse relations problems. The type of reasoning that could, but did not necessarily, lead to the correct answer is displayed in bold. For the lowest ability group (4 year olds) inverse problems, as expected, proved too difficult, with just under half of the children's justifications (.46) classified as unclear. As already observed in relation to direct relations, children at this ability level were more likely to reason in terms of the similarities between drinks than in terms of the differences. At the next ability level (5–6 year olds), there is more evidence of children using inverse reasoning (.20 and .35), but as the proportion of direct, inverse and same taste justifications are similar at this ability level it becomes apparent why the Rasch model places inverse items as being too difficult for these children. This indecisive pattern of justifications is also present for children in the third ability grouping (7 year olds), which is again unsurprising in terms of this group's placement along the Rasch model. In terms of justifications it is only at the highest ability level of the current sample (8 and 9 year olds) that inverse reasoning is used in a more systematic way. These justifications again confirm the predictions of the Rasch model that children between 8 and 9 years should be able to handle inverse relations problems.

Table 3.11 Proportion of Justifications Used by Children at Different Ages to Solve Problems of Inverse Relations

Ability group	Age	Unclear	Same drink	Controlled variable	Direct	Inverse	Proportion
1	4	.46	.03	.22	.16	**.13**	-
2	5	.20	.04	.25	.30	**.20**	.01
	6	.07	.02	.30	.27	**.35**	-
3	7	.13	.03	.22	.29	**.32**	-
4	8	.11	.01	.10	.20	**.58**	.01
	9	.04	.01	.09	.26	**.60**	-

3.7.5.6 Children's Justifications by Age to Sampling Problems

Table 3.12 presents what proportion of the different age groups offered each type of justification about their judgements made for sampling items. The type of reasoning that could, but did not necessarily, lead to the correct answer is displayed in bold. Here, the lowest ability group (4 year olds) use direct relations (.41) to justify their judgements more than any other strategy. The justifications of the second ability group (5 and 6 year olds) shows a movement away from incorrectly applying direct relations and more emphasis on strategies that could lead to the correct solution, with one-fifth of the justifications of 6 year olds based on the idea that pouring a drink does not alter its taste. Around a quarter of children at this ability level make the 'same sugar' mistake by claiming that both the sampled and un-sampled drinks will contain the same absolute quantity of sugar. For example, a drink containing one lump of sugar when poured into two separate glasses will produce two new drinks, each containing one lump of sugar.

The third ability grouping (7 year olds) appears to approach the problem of sampling with a strategy similar to that of the 5 and 6 year olds, also employing the 'same sugar' idea. As the model viewed the sampling items as being too difficult for these children it is perhaps not surprising that the problem-solving strategies they applied to the situation were quite similar. The highest ability grouping (8 and 9 year olds) produced mean ability scores close to that required to pass the sampling problems. The explanation of this finding is that children increasingly used proportional reasoning justifications, or stated that pouring does not affect the taste of the drink. The sampling problem was still difficult for these children, with the majority of erroneous justifications referring to direct or inverse relations.

Table 3.12 Proportion of Justifications Used by Children at Different Ages to Solve Problems of Sampling Taste

Ability group	Age	Unclear	Same drink	Direct	Inverse	Proportion	*Original state*	*Same sugar*	*No sugar*
1	4	.38	-	.41	.03	-	***.03***	*.15*	-
2	5	.29	**.04**	.22	.09	**.06**	***.02***	*.26*	*.02*
	6	.07	**.07**	.17	.03	**.17**	***.20***	*.30*	-
3	7	.27	-	.18	.12	-	***.09***	*.35*	-
4	8	.04	-	.21	.11	**.11**	***.27***	*.12*	*.14*
	9	-	**.04**	.33	.28	**.13**	***.17***	*.02*	*.02*

3.7.5.7 Children's Justifications by Age to Proportional Equivalence Problems

Table 3.13 summarises what proportion of the different age groups offered each type of justification about their judgements on items of proportional reasoning. The type of reasoning that could (but did not necessarily) lead to the correct answer is displayed in bold. For proportional reasoning problems the first three ability groups made little progress, as practically all the children at these ages reasoned in terms of direct or inverse relations, that is, on the importance of one quantity only. The highest ability grouping (8 and 9 year olds) also showed a preference for this type of one-quantity reasoning, but in addition to this, children of these ages also attempted proportional reasoning on a larger scale.

Table 3.13 Proportion of Justifications Used by Children at Different Ages to Solve Problems of Proportional Reasoning

Ability group	Age	Unclear	Direct	Inverse	Proportion
1	4	.21	.44	.35	-
2	5	.13	.59	.24	**.04**
	6	.07	.33	.60	-
3	7	.06	.53	.35	**.06**
4	8	.08	.33	.29	**.30**
	9	-	.33	.35	**.33**

3.7.6 Summary of the Results of Experiment 2

The results of the Rasch analysis provided a general confirmation that Strauss and Stavy's (1982) original ordering of logical relations regarding sweetness extends to the broader intensive quantity concept of taste. This led to the acceptance of the experimental hypothesis. The analysis of justifications considered Strauss and Stavy's (1982) reports of a relation between inverse relation success and an understanding of proportional equivalence. Children in the highest ability group (8 and 9 year olds) displayed a marked increase in understanding the proportion of inverse relations and proportional equivalence justifications compared to the other three ability groups. This indicates that competence with inverse relations leads to increased competence on proportional reasoning problems.

3.8 Discussion of Study 1

The aim of Study 1 was to revisit some of the findings reported within the proportional reasoning literature to see how they influence performance on direct and inverse relations problems. Study 1 was reported as two experiments: Experiment 1 attempted to answer the question of methodologies, whereas Experiment 2 looked at the links between direct and inverse reasoning and other important logical ideas of the intensive quantity concept.

Of specific interest during Experiment 1 was whether using computer diagrams to display problems – the proposed methodology for further experiments conducted in this thesis – provides a fair reflection of children's performance on relational thinking problems when compared to more concrete and tangible forms of presenting a mathematical problem. The need to consider this question comes from previous work on sampling by Schwartz and Moore (1998), who showed that mode of presentation significantly affected children's performance. The results from Experiment 1 showed that children receiving physical demonstrations of direct and inverse relations problems did not perform significantly differently from children receiving computer diagrams.

Therefore, during further studies in this thesis computer diagrams can be used as an appropriate methodology to look at children's understanding of direct and inverse relations with intensive quantities. The second methodological question considered during Experiment 1 was whether presenting problems with manipulatives improves performance on direct and inverse relations problems. Tourniaire and Pulos (1985) reported that manipulatives improve the task performance of children's with low proportional reasoning skill. The experiment tested whether this benefit is observable on direct and inverse relations problems. Although children receiving manipulatives performed significantly better on direct relations items when compared to children receiving physical demonstrations, this was a very specific advantage and an advantage was not observed between the performance of children receiving manipulatives over that of children receiving computer diagrams.

The third issue from the literature addressed in Study 1 regarded the links between direct and inverse relational thinking and the wider intensive quantity concept. A second experiment addressed this issue. Here two observations from the original study of logical relations by Strauss and Stavy (1982) were tested in intensive quantity settings. The first observation tested was the reported order of difficulty of different logical relations. The second observation tested was the link reported between inverse reasoning and proportional equivalence. As Strauss and Stavy's observations were only recorded within the setting of a specific quantity ('sweetness'), Experiment 2 extended this to cover the intensive quantity of taste. The experimental hypothesis tested whether:

'In taste settings, the order of relative difficulty of different logical relations will follow Strauss and Stavy's (1982) observations in the context of sweetness from direct relations, inverse relations, sampling, and proportional equivalence.'

The results show that the four relations important to the intensive quantity concept scaled along one dimension, with an order of difficultly consistent with that reported by Strauss and Stavy (1982). The results of the second experiment also highlighted the ages of 7–9 years as the time when children make the most significant progress with inverse relations problems.

Having established a suitable methodology and an age range of particular interest, the question moves from looking at different task variables to considering the question of conceptual difficulty. The question of conceptual difficulty is pursued more specifically with a comparison of direct and inverse relations in intensive and extensive quantity settings in Study 2.

CHAPTER 4

STUDY 2: A COMPARISON BETWEEN CHILDREN'S UNDERSTANDING OF EXTENSIVE AND INTENSIVE QUANTITIES

4.0 Aim

There are two aims for study 2. The first aim is to test whether children's difficulties with intensive quantity non-computational problems can be explained by the need to consider inverse relations. To achieve this children's success on direct and inverse relations problems is compared across extensive and intensive quantity contexts. The second aim is to analyse children's difficulties across a wider range of intensive quantity problems than those analysed in Study 1.

4.1 Background

4.1.1 Can Children's Difficulties with Intensive Quantity Comparison Problems be Explained by a General Problem of Inverse Relations?

Study 1 established that children solved direct relations intensive quantity problems more easily than the inverse problems. It was also shown that between the ages of 7 and 9 years children made significant progress in understanding the inverse relation. This leads to the question of whether a general difficulty with understanding inverse relations can explains children's difficulties with intensive quantity non-computational problems. As the problems designed for Study 1 were all presented within a specific intensive quantity setting (taste), Study 2 tests this possibility by comparing children's success with intensive quantity non-computational problems against a comparable set of extensive quantity non-computational problems. This will provide a direct test of whether intensive quantities represent a unique challenge for children, above and beyond the difficulty of understanding inverse relations between quantities.

The design of the current study follows that of Vergnaud, Rocchier, Ricco, Marthe and Metregiste's (1978) study of multiplicative structures in which they tested whether an important distinction exists between product of measures and multiple proportions problems. Vergnaud et al (1978) compared the success of 11–12 year olds asked to solve product of measures problems based on the calculation of volume, and multiple proportions problems for which children had to calculate milk yields by considering the relations between three independent quantities (number of cows, litres of milk, days). By designing problems in which the number of variables in each problem structure was the same, the two could be compared. Vergnaud et al. were able to show that with direct relations problems, children were more successful on multiple proportion problems than on product of measures problems. The comparison of extensive and intensive non-computational problems in the current study adopts the approach taken by Vergnaud et al's study while expanding the original scope of this design to compare problems that involve both direct and inverse relations.

4.1.2 Testing Children's Understanding of Intensive Quantities Across a Wider Range of Settings

The second aim of the current study is to examine children's relational thinking across a wider range of intensive quantity settings than were used in Study 1. The question of interest here is whether children experience the same types of difficulties between intensive quantity settings. As the problems for the current study are designed as non-computational, multiple choice problems, difficulties can be represented by two types of error. The first type of error would be to only attend to one quantity, while the second would be to regard both quantities as important to the intensive quantity, but to struggle with the idea of inverse relations. In this instance children would reason 'the more A, the more B' regardless of whether the quantities are directly or inversely related to the intensive quantity. This second type of erroneous reasoning strategy has been observed elsewhere in children on the verge of solving inverse extensive quantity sharing problems

(Correa, Nunes, and Bryant, 1998; Squire and Bryant, 2003). The current study extends the analysis of errors adopted in these sharing experiments to consider the errors of children shown from Study 1 to be on the verge of solving inverse intensive quantity problems.

4.1.3 Rationale

In order to carry out the investigation, a series of non-computational problems was designed. For these non-computational problems, one of the variables is controlled, so that the children only need to consider the relation between the second variable and the quantity about which they are asked. For example, if the children are asked 'which drink is sweeter?' either the amount of sugar stays the same and the volume of liquid varies, or the volume of liquid stays the same and the amount of sugar varies.

All problems involve three variables. In some of the problems, all the variables are extensive quantities (e.g. it is possible to establish relations between the following three extensive quantities; number of animals, number of bags of food, and the number of days the food will last) whereas in other problems there are two extensive quantities and one intensive quantity; the target quantity in the problem is the intensive quantity. Thus, the number of variables involved in the problems is always the same but the type of quantity varies. Within each type of quantity, half of the problems require reasoning about a direct proportional relation whereas the remainder require reasoning about an inverse proportional relation.

There are three experimental hypotheses. The first and second hypotheses relate to testing the distinction between extensive and intensive quantity problems and direct vs. inverse relations. The third hypothesis was generated to investigate whether the reasoning difficulties children experience with inverse intensive quantity problems are consistent with those reported in the sharing literature on extensive quantities where children on the verge of solving inverse relations over apply 'the more A, the more B' responses.

4.1.4 Hypotheses

Hypothesis 1:

'Controlling for type of relation, intensive quantities will be significantly more difficult than extensive quantity problems'

Hypothesis 2:

'Problems about inverse relations will be significantly more difficult than those about direct relations, irrespective of the type of quantity'

Hypothesis 3:

'Children's errors with inverse intensive quantity problems will be explained by the overuse of the 'the more A, the more B' reasoning strategy'

4.2 Method

4.2.1 Participants

A total of 113 children (50 boys and 63 girls) in Years 3 and 4 in three different primary schools in Manchester participated in this study. The age range was from 7 years 8 months to 9 years 7 months; the mean age was 8 years 6 months.

4.2.2 Design

The study employed a mixed model 2×2×2 design: quantity type (extensive vs. intensive – within participants) by type of relation (direct vs. inverse – within participants) by year group in school (Year 3 vs. Year 4 – between participants) design.

4.2.2.1 Experimental Task

There were four problems about extensive quantities, two involving direct and two involving inverse relations; there were 18 problems about intensive quantities, nine involving direct and nine involving inverse relations. There were a larger number of problems about intensive quantities in order to allow the exploration of a wider range of intensive quantities to test the second aim. Four further problems about additive reasoning were included as distracters.

As with Study 1 the problems were designed using the language controls first reported in Desli (1999). Because intensive quantity problems always involve direct and inverse relations, language controls are needed so that significant differences can be attributed to the direction of relation and not the arrangement of variables in the problem. This point is illustrated by the sample problem given in figure 4.1. Here two different cost situations are depicted; in the first, the size of the product is controlled while the price to be paid is varied. In the second problem, the size of the product is varied while the price to be paid is controlled. In the first variation (product size controlled, price varied), asking children whether one product is more expensive than the other requires the children to understand the direct relation of 'more money' means 'more expensive'. Using the linguistic form 'cheaper' this problem then becomes an inverse relations problems – less money to pay means the product is cheaper. In the second variation of the problem (product size varied, price controlled) the 'more expensive' linguistic form leads to the inverse relation – less product is more expensive – while for the 'cheaper' form the relation is direct – more product means it is a cheaper purchase. By introducing these types of language controls it is possible to consider the difficulty of relational thinking independently from the problem setting. It must also be noted that in the setting of cost, three different verbal formulations were used: 'cheaper', 'more expensive', and 'better value'. This last variation was used to test whether expressions 'cheap' and 'expensive' were too closely connected to price in everyday life and could perhaps mislead the children to ignore the quantity purchased; the expression 'better value' was used in order to assess this possibility. It was expected that when considering the offers which are 'better value', both product size and money paid are important.

Figure 4.1 presents two intensive quantity problems from the study and Figure 4.2 presents samples of two extensive quantity problems from the study, showing how the language was controlled across the direct and inverse proportions questions. Appendix 4.1 lists all the problems in this study.

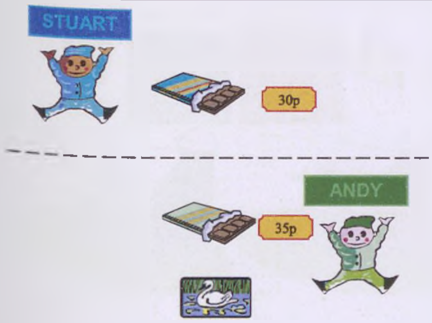
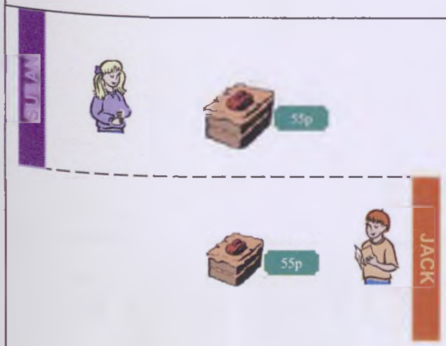
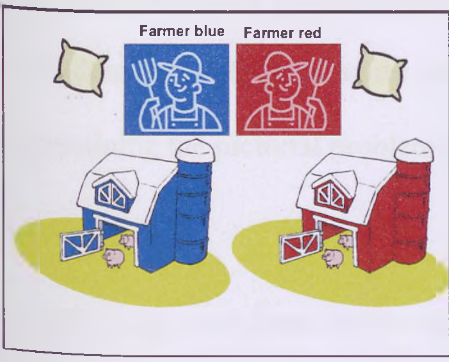
	Cost: Product controlled/Price varied Stuart bought a big bar of chocolate from the sweet shop; it cost 30p. Aaron bought a big bar of chocolate in another sweet shop; it cost 35p. Was the chocolate better value in one shop than in the other shop? Circle yes or no. If you circled yes, write the name of the person whose chocolate was better value.
	Cost: Product varied/Price controlled Susan bought a big cake from the cake shop; it cost 55p. Jack bought a small cake in another cake shop; it cost 55p. Was the cake better value in one shop than the other? Circle yes or no. If you circled yes, write the name of the person whose cake was better value.

Figure 4.1 Examples of intensive quantity problems

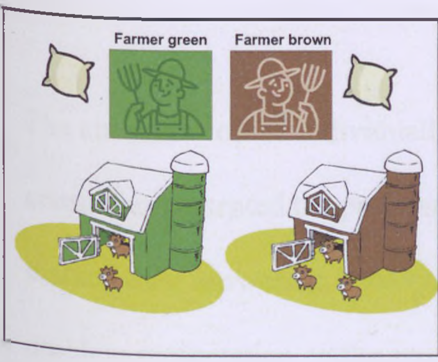


Direct question

Farmer Blue (on the right) and Farmer Red (on the left) keep the same number of pigs on their farms. They both feed their pigs the same amount of food every day.

Farmer Blue has 20 bags of pig food and farmer Red has 15 bags of pig food.

Will one Farmer be able to feed his pigs for more days before he has to go shopping for more? Circle yes or no in your booklet. If you circled yes, write the name of the farmer who will be able to feed his pigs for more days.



Inverse question

Farmer Green and Farmer Brown both keep cows on their farms. Farmer Green has 10 cows and Farmer Brown has 15 cows. Farmer Green and Farmer Brown have both just bought 30 bags of cattle food each from the market.

They both feed each of their cows the same amount of food every day.

Will one farmer be able to feed his cows for more days? Circle yes or no in your booklet. If you circled yes, write the name of the farmer who will be able to feed his cows for more days.

Figure 4.2 Example of extensive quantity problems

4.2.3 Materials

A laptop computer and data projector displayed the problems in pictorial form to the children during the task administration. The children recorded their answers in booklets containing the pictorial problems as displayed by the computer.

4.2.4 Ethical Approval

Ethical approval from the Social Sciences and Law Ethics Committee at Oxford Brookes University was obtained: An initial letter was sent to the head teachers of all the schools involved (Appendix 4.2). This letter contained a suggested letter of consent to be sent out to the parents of the children (Appendix 4.3). For each school the head teacher acted as the gatekeeper and was free to decide whether to seek permission from parents for their child's participation in the study.

4.2.5 Procedure

The children worked individually during a whole-class presentation of the problems. The researcher presented the problems on a screen at the front of the class and the children marked their answers in booklets, where each page contained the same picture used for problem presentation on the screen.

The researcher started the session with a brief discussion of what research is and how important the children's contribution to this research would be. They were asked to work individually throughout the session. The class teacher and the researcher ensured that this was observed. After projecting each picture on the screen, the researcher gave the instructions orally. The children were given time to complete their answer. After all the children indicated that they had completed the answer, the next item was presented. At the end of the session, the children were given an opportunity to ask questions and were thanked for their cooperation.

4.3 Results

The results section is presented in two parts. The first part presents the tests of main effects. The analysis of main effects is used to pursue the first aim and test hypotheses 1 and 2 to determine whether the distinction between intensive and extensive quantities (Hypothesis 1) and direct and inverse relations (Hypothesis 2) is important for non-computational problems. The second part of the results focuses on children's performance on intensive quantity problems. The data in this section is used to test Hypothesis 3 about whether children's problems with intensive quantity problems can be explained by an overuse of the direct relation 'the more A, the more B'.

4.3.1 Test of Main Effects

The proportion of correct responses for each problem type is presented in Table 4.1. The analysis of the skewness of the distributions showed that, although the performance of 8 year olds was very good on direct relations intensive quantity items, this distribution was not significantly skewed ($z=1.623$). So it was possible to use analysis of variance to assess the effect of the different factors on the children's performance. There were different numbers of intensive and extensive quantity problems, so the proportion rather than the number of correct responses was used in this analysis. Before carrying out the analysis, an Arcsine transformation was applied to the proportion of correct responses as recommended by Ferguson (1971).

Table 4.1 Proportion Correct and Standard Deviation for Each of the Four Problem Types by School Year (N=113)

Problem Type	Direct relation Mean *(SD)*	Inverse relation Mean *(SD)*
Year 3 (7 year olds) N=82		
Extensive Quantities	.77 *(.31)*	.60 *(.37)*
Intensive Quantities	.61 *(.17)*	.34 *(.15)*
Year 4 (8 year olds) N=31		
Extensive Quantities	.73 *(.34)*	.65 *(.35)*
Intensive Quantities	.63 *(.18)*	.40 *(.18)*

As can be seen in Table 4.1, children in both year groups scored higher on direct than inverse problems regardless of whether the problems involved extensive or intensive quantities. It can also be seen that for both year groups the extensive quantity problems which the children found the most difficult (inverse relations) were solved with comparable rates of success as the intensive quantity problems with which the children were the most successful (direct relations).

For the analysis of main effects Mauchly's Test of Sphericity was shown not to be significant which means unadjusted F scores are reported. Levene's test of equal variances was also not significant meaning that equal variances could be assumed. The analysis of variance showed a significant effect of type of quantity ($F_{1,111}$=80.24, p<.0001) with a large Cohen's d effect size of 0.76 and a significant effect of type of relation ($F_{1,111}$=40.63, p<.0001) with a medium Cohen's d effect size of 0.51. There was no significant effect of School Year ($F_{1,111}$= 0.22, p=.63). None of the interactions was significant. This means that it is possible to accept Hypothesis 1 which predicted a main effect of quantity, and Hypothesis 2 which predicted a main effect of relation. It can be concluded from these results that the difficulty of reasoning about inverse proportional relations does not account for the difficulty of understanding intensive quantities. Reasoning about inverse proportional relations in intensive quantity settings is significantly more difficult than reasoning about inverse proportional relations in extensive quantity settings.

4.3.2 Children's Performance on Intensive Quantity Problems: General Performance

To test the third hypothesis, children's levels of success between intensive quantity settings were considered. If children were over-applying direct relational thinking across intensive quantity problems then it would be expected that children would solve the direct form of the problem more easily than the inverse form. The proportion of correct responses in each intensive quantity problem is presented in Table 4.2.

Table 4.2 Proportion of Correct Responses for Each Type of Relation Controlling for the Language Used in the Question

	Direct Relations Question		Inverse Relations Question	
	Proportion	*(SD)*	Proportion	*(SD)*
Which flowerbed will be more crowded?	.86	*(0.35)*	.56	*(0.50)*
Which flowerbed will be less crowded?	.73	*(0.44)*	.44	*(0.50)*
Which drink will be sweeter?	.82	*(0.38)*	.26	*(0.44)*
Which drink will be sourer?	.58	*(0.49)*	.19	*(0.39)*
Which one was more expensive?	.70	*(0.46)*	.13	*(0.33)*
Which one was cheaper?	.73	*(0.44)*	.13	*(0.34)*
Which purchase was a better value?	.35	*(0.48)*	.64	*(0.48)*
Which car travelled more quickly?	.11	*(0.31)*	.81	*(0.39)*
Which car travelled more slowly?	.66	*(0.47)*	.16	*(0.37)*

Table 4.2 shows that when the language is controlled, the direct proportions problems were considerably easier than the inverse proportions problems in seven of the nine items. Only

two problems did not follow this pattern, one of them related to cost and the second related to speed. The problem about cost that did not follow the pattern of direct questions being easier than inverse questions asked the question 'which purchase was a better value?' This linguistic form was introduced because it was expected to make it easier for the children to consider not only the price paid but also the amount purchased in the problems. If this linguistic format helped the children to consider the amount purchased, this would produce better results for this question than the previously used formats, 'which was cheaper?' and 'which was more expensive?'. However, this does not seem to be the case for the overall proportion correct, although there is an increase in the proportion correct for the inverse question, and there is a decrease in the proportion correct for the direct question. Because the direct question in this case required a consideration of the amount purchased – the more bought for the same amount of money, the better the value – the decrease in the proportion correct suggests that the children continued to focus more on price paid rather than on amount purchased. Thus the change in linguistic format had little influence on the results: the children seem to consider one variable (price) more than the other (quantity purchased). It is quite possible that the connection between low price and good purchase in everyday life is so strong that other variables which might make one purchase better than another become obliterated from children's analysis.

Similarly, the problems related to speed suggest that the children consider one variable (time) more than the other (distance). When the question asked was 'which one travelled more quickly?', the shorter the time, the more quickly the car travelled. This made the inverse question easier than the direct question, where distance was not treated as relevant. When the question asked was 'which one travelled more slowly?', the shorter the distance, the more slowly the car travelled (inverse relation) and the longer the time, the more slowly the car travelled (direct relation). Of these, the children solved the direct form more easily. So the children were able to think about direct and inverse relations between time and the expressions 'quicker' and 'slower' but they failed to consider the distance travelled. This renders the problems where time is controlled more difficult than those where distance is controlled. Therefore, cost and speed appear to be genuine exceptions to the rule that direct relations problems are easier.

4.3.3 Children's Performance on Inverse Intensive Quantity Problems: Error Analysis

The second way in which the reasoning pattern 'the more A, the more B' could appear is in the reasoning of children who have difficulty with the inverse relation as was shown in the sharing of extensive quantities experiments (Correa, Nunes, and Bryant, 1998; Squire and Bryant, 2003). To test for systematic direct relations errors on inverse relations problems, the children's errors were classified into two categories. The first category captured the 'the more A, the more B' error, while the second category captured 'the same A, the same B' responses – for example, if the amount of sugar is the same in each drink then the drinks will taste the same (correct responses were excluded from the analysis). Chi-square tests were carried out on the frequency of errors of each type to test whether systematic errors appeared in each intensive quantity setting. Table 4.3 presents the proportion of errors by type for each inverse relations setting. In the table, the bold indicates significant chi-square statistics, $p<.0001$; the proportions do not add to 1.0 because some responses were correct.

Table 4.3 Proportion of Error Types by Question for the Inverse Problems

	The intensive quantities are the same	'The more A, the more B' error
Which flowerbed will be more crowded?	.26	.15
Which flowerbed will be less crowded?	.25	.28
Which drink will be sweeter?	**.52**	**.21**
Which drink will be sourer?	**.02**	**.76**
Which one was more expensive?	**.76**	**.07**
Which one was cheaper?	**.77**	**.13**
Which purchase was a better value?	**.05**	**.26**
Which car travelled more quickly?	**0**	**.17**
Which car travelled more slowly?	**.75**	**.16**

Two observations are immediately apparent from the data in Table 4.3 which suggest that children's difficulties with inverse relations problems are more complex than the over-application of direct relations. The first is that systematic errors were found in three of the four intensive quantity settings. The exception was crowdedness, for which children did not make significantly more errors of either type. The second observation is that although children made systematic errors in the remaining intensive quantity settings, the types of errors made were not always due to the over-application of the direct relation. For example with taste, the linguistic form 'sweetness' produced significantly more 'the same A, the same B' errors while 'sourness' produced significantly more 'the more A, the more B' errors. As the consistent finding of 'the more A, the more B' errors did not occur between problem settings, Hypothesis 3 was rejected. The analysis of children's performance on intensive quantity problems shows that, when constructing an understanding of intensive quantities, children have to face two challenges: firstly, to think in terms of proportional relations and secondly, to understand the connection between the intensive quantity and the two extensive quantities which are related to it.

4.4 Conclusion and Discussion

4.4.1 Can the Difficulty of Intensive Quantity Non-Computational Problems be Explained by a Problem with Thinking About Inverse Relations?

The first aim of Study 2 was to test whether the difficulty with intensive quantity non-computational problems was due solely to the fact that they involve inverse relations. For type of quantity the data analysis showed a clear main effect: intensive quantity problems were significantly more difficult than those composed only with extensive quantities. This means that it was possible to accept the first experimental hypothesis which stated that controlling for type of relation, intensive quantity problems will be more difficult than extensive quantity problems. This pattern of results showed that the difficulty of understanding intensive quantities could not be accounted for by the difficulty of thinking about inverse proportional relations. The children were significantly more successful on items that required thinking about inverse relations when all the quantities in the problem were extensive than when the target quantity was an intensive quantity. Thus, intensive quantities must encompass an added difficulty.

On the question of direct vs. inverse relation problems, generally speaking direct relations problems were easier than inverse relations problems as shown by the significant main effect of relation. Although there were occasions when children scored quite highly on an inverse relations problem and lowly on the related direct relations problem, in terms of the general test of direct vs. inverse relations these variations could be regarded as an issue of error of measurement. As Hypothesis 2 alludes to children's general performance, it is possible to accept this hypothesis which stated that problems about inverse relations will be more difficult than those about direct relations, irrespective of the type of quantity. By extending Vergnaud et al's (1978) original design to compare direct and inverse relations, it was possible to show how direction of relation represents an important conceptual challenge between multiplicative structures.

4.4.2. Testing Children's Understanding of Intensive Quantities Across a Wider Range of Settings

The second aim of the Study 2 was to look at children's intensive quantity understanding across a wider range of settings. The reason for extending the number of intensive quantity setting under investigation was to see whether children experienced a common difficulty across intensive quantity problems. It was expected that findings reported in the literature on children's intuitive rule use would mean that children would make systematic 'the more A, the more B' errors on inverse relations problems.

The analysis of the children's performance between intensive quantity settings showed that in most settings the direct form of the problem was easier to solve than the inverse form, although for problems involving speed and cost this was not always the case. In these settings, children performed better on the inverse problem than on the direct problem when the inverse relation involved time and money variations respectively. Further analysis of children's errors on inverse relations problems showed that although children did make 'the more A, the more B' errors in some settings, 'the same A, the same B' errors were observed more frequently in others. This led to the rejection of the third experimental hypothesis which stated that children's errors with inverse relations intensive quantity problems could not be explained by the overuse of 'the more A, the more B' reasoning strategy, as predicted by intuitive rule theory (Stavy and Tirosh, 2000).

This analysis of errors showed that relational thinking about intensive quantities encompasses the initial added difficulty of considering the role of the second variable in the problem. This finding supports the work on sharing (Correa, Nunes, and Bryant, 1998) which found that younger children failed inverse relations problems by reasoning inconsistently about the variables in the problem while older children's on the verge of solving inverse relations problems make more systematic direct relations errors. The current findings extend Correa et al's findings to intensive quantity problems. There are two issues to be examined in the next study, firstly whether these results can be replicated with an older sample, and secondly, whether children's performance can be influenced by varying the way in which the quantities in the problem are represented?

CHAPTER 5

STUDY 3 THE EFFECT OF REPRESENTING PROBLEM VARIABLES NUMERICALLY ON CHILDREN'S UNDERSTANDING EXTENSIVE AND INTENSIVE QUANTITIES

5.0 Aim

There are two aims to Study 3. The first aim is to replicate the two main findings obtained in Study 2 which were that children performed significantly better on extensive quantity problems than intensive quantity problems, and significantly better on direct relations problems than inverse relations problems. The second aim of Study 3 is to test whether the use of numerical information significantly affects children's reasoning in these problems, even though they are not asked to carry out any computations.

5.1 Introduction

5.1.1 Replicating the Findings of Study 2

Study 2 showed that after controlling the number of quantities in the problem, quantity type (extensive or intensive) determined the relative difficulty of direct and inverse relational thinking problems. With no replications currently in the literature, a replication study is desirable to strengthen the external validity of this result. The sample participating in Study 2 were between 7 and 9 years old with the majority of children aged 7 years. To consider the extent to which the distinction between extensive and intensive quantities is important when children are expected to be using relational thinking more systematically, the age range in Study 3 is expanded to include a larger sample of 9 year olds.

5.1.2 Presentation of Problems with Numerical or Relational Information

In addition to replicating the general design of Study 2, Study 3 introduces a distinction between the presentation of the information essential to problem solutions. For the non-computational problems currently under investigation there are two possible ways in which problem information can be presented across all intensive quantity settings. The first way is to represent the quantities in the problem numerically, the second way is to substitute numerical information for relational terms such as 'more', 'less', and 'same'. The effect of the numerical presentation of quantities on children's reasoning has received significant attention in the literature on joining and sampling intensive quantities, though the significance of this variation on direct and inverse non-computational problems remains largely unexplored.

Studies looking at the joining of intensive quantities (Cowan and Sutcliffe, 1991; Desli, 1999) asked children to predict the final temperature when mixing two glasses containing liquids of identical or varying temperatures. In some instances, the Celsius scale showed the pre-mixing temperatures of the liquids (e.g. '20ºC' and '20ºC'). On other occasions, children received qualitative descriptions of the pre-mixing temperatures (e.g. 'warm water' and 'warm water'). A consistent finding of both studies was that giving children the temperature in Celsius produced more additive errors. Desli found that this error also extended to paint-mixing tasks. Both studies reported that children asked to make qualitative comparisons used correct averaging solutions more frequently. Ahl, Moore and Dixon (1992) also found that with joining of intensive quantities problems, children who received problems with qualitative information first went on to solve problems with numerical information more successfully than children receiving the numerical information problems first.

Turning to the literature on sampling, in a finding consistent with work on the joining of intensive quantities, Harel, Behr, Lesh and Post (1994) reported that children aged 11–12 years solving sampling problems with numerical information made more mistakes:

'What is amazing about the response is that for AN, whether the glasses would taste the same or not depended on whether she knew the actual sizes of the glasses and the amounts

of the ingredients. This may have to do with AN's belief about mathematics that when numbers are involved in a problem, computation must be carried out to solve the problem.' (p.340)

Later work by Schwartz and Moore (1998) also suggested that the presence of numerical information influenced children's reasoning. Working with 11–12 year olds, they looked at how varying problem presentation and numerical information affected performance on sampling tasks. Presenting sampling of orange juice problems, Schwartz and Moore compared performance between types of presentation – photographs and diagrams (everyday vs. mathematical context) – when numbers were either present or absent from the problem. It was shown that children's performance on problems presented with photographs was comparable regardless of whether or not the problems contained numerical information. For children receiving problems with diagrams, the presence of numerical information resulted in significantly lower task performance. Schwartz and Moore concluded that when problems were presented as diagrams, children took a more analytical stance to the problem. This meant that when numerical information accompanied the diagrams, children tried to use numerical comparisons beyond their current understanding of proportional relations. It was argued that numerical information did not have the same impact in the photographic condition because the photographs acted as a cue to empirical knowledge of the problem which children used because they were unsure how to reach the solution using the numerical information.

The literature on joining and sampling intensive quantities demonstrates that numerical information can significantly affect children's thinking. The mere presence of numbers in joining tasks led to erroneous additive solutions, whereas in sampling studies the relationship between numerical information and problem difficulty was less clear-cut. The interesting point from these studies was that children do not need to perform computations to solve sampling and simple joining problems. With sampling problems in particular children do not even have to think about the numbers in the problem being solvable with what Harel, et al.(1994) refer to as an 'intuitive internalised ratio' (p.327).

The issue of representing quantities numerically in direct and inverse relations problems is different. Direct and inverse relations problems are similar to sampling and simple joining of intensive quantity problems in that these problems can be solved without the need for computations. However, there is an important difference in that with the most basic direct and inverse relations problems, numerical information can provide important clues as to the quantities which children need to compare.

The conclusion reached in Study 2 was that to understand relational thinking with intensive quantities, children had to first understand that both quantities were important (in a similar manner to Piaget's 1946 original conservation experiments) to the intensive quantity before thinking about the direction of the relation. There are two ways in which the presence of numerical information may impact on children's relational thinking. Firstly children's performance on direct and inverse relations problems could mirror the reports of studies from the sampling and joining of intensive quantities literature. In this instance the presence of numerical information would lead children to attempt additive comparisons. If the presence of numerical information leads children to additive comparisons then ceiling level performance should be observed on direct relations problems. It would also be expected that an additive bias would lead to lower levels of performance on inverse relations problems, when compared to children receiving relational information would be expected.

A second possibility is that numerical information cues children to the importance of both the quantities which comprise an intensive quantity. This would lead to significantly better performance in children receiving problems when compared to children receiving relational information only. Observations of children solving inverse relations problems presented with numerical information earlier than the general proportional reasoning literature suggested can be found in Noelting's (1980) classic proportional reasoning study. Noelting presented children with mixing of orange concentrate and water problems in which both orange concentrate and water represented numerical by a certain number of glasses. Using this problem format Noelting's descriptions of children's reasoning and success differed in two ways from those described in Study 2. Firstly Noelting's study reports children solving inverse relations problems systematically at around 6 ½ years. Secondly, Noelting's descriptions of children's solution strategies, include no instances of children incorrectly applying direct relations to inverse relations problems. Noelting reported that children in his 'first stage' understood the role of the orange concentrate and ignored the quantity of water, before his 'second stage', in which children also paid attention to the second quantity (water) leading to the passing of inverse relations problems.

It must be noted that in Noelting's study children completed sample problems in which they were asked to explicitly think about the role of water in the problems. Therefore, even if numerical information indicates the importance of considering both quantities to the children, the presence of numbers offers no indication as to which direction the relation between intensive quantities and their extensive quantity referents takes.

5.1.3 Hypotheses

Study 3 has three experimental hypotheses, hypotheses 1 and 2 relate to the replication of the results observed in study 2. The third experimental hypothesis was formulated to follow the predictions of the intensive quantity literature on sampling and joining that relational information would lead to improved performance on inverse relations problems.

Hypothesis 1:

'Children will score significantly higher on extensive quantity problems when compared with intensive quantity problems.'

Hypothesis 2:

'Direct relations problems will be significantly easier than inverse relations problems.'

Hypothesis 3:

'Children receiving problems with qualitative information will score significantly higher on inverse relations problems than children receiving the same problems with numerical information.'

5.2 Method

5.2.1 Participants

A total of 244 year 3 and 4 children (133 boys and 111 girls) participated from five Oxfordshire primary schools and one London primary school. The mean age of the children was 8:9 years with a standard deviation of 7.0 months with a range of 7:9 to 9:9 years.

5.2.2 Design

Study 3 employs a mixed model 2x2x2x3 design: quantity type (extensive vs. intensive – within participants) by relation type (direct vs. inverse – within participants) by presentation of information (quantitative vs. relational comparison – between participants) by age category (7 years vs. 8 years vs. 9 years – between participants).

The children within each class tested were randomly assigned into either the quantitative or the relational condition. The children also completed a control task of general mathematical reasoning to verify that no ability differences existed between the experimental groups.

5.2.2.1 Experimental Task

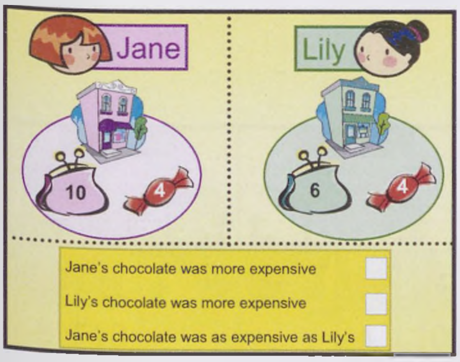
The experimental task consisted of 20 problems delivered orally by a researcher during a single whole-class session. Eight problems involved intensive quantities (four direct and four inverse relations), and eight problems involved extensive quantities (four direct and four inverse relations). Four additive distracter items using similar language to the target problems were also included. Figure 5.1 and Figure 5.2 show example problems from the quantitative and relational conditions. For each school the problems were randomly ordered, with the exception that the first item was always a direct relations taste problem, which the previous studies had indicated were easy for children of this age group.

Presenting an easy item first helped the children settle into the test situation. The full set of problems is available in Appendix 5.1.

5.2.2.2 Quantitative Condition

Children assigned to the quantitative condition received full numerical information for two variables in the problem from which to compare a third intensive or extensive variable. For an example of this question format for extensive and intensive quantity problems, see Figure 5.1.

Intensive direct quantitative condition



Jane has 10p, Lily has 6p.

Jane buys 4 chocolates using all her money

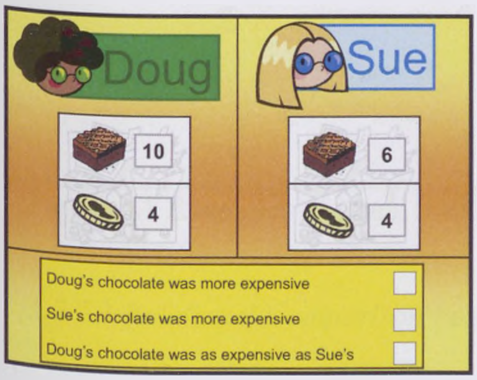
Lily buys 4 chocolates using all her money.

Were Jane's chocolates more expensive?

Were Lily's chocolates more expensive?

Or were Jane's chocolates as expensive as Lily's?

Intensive inverse quantitative condition



Doug buys 10 chocolates.

Sue buys 6 chocolates.

Doug spent 4p and Sue spent 4p.

Were Doug's chocolates more expensive?

Were Sue's chocolates more expensive?

Or were Doug's chocolates as expensive as Sue's?

Extensive direct quantitative condition

Farmer Blue has 4 bags of sheep food

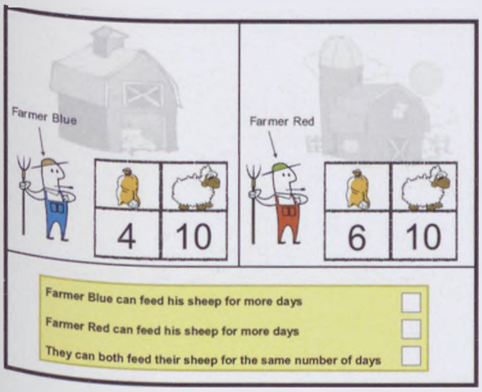
Farmer Red has 6 bags of sheep food.

Farmer Blue has 10 sheep and Farmer Red has 10 sheep.

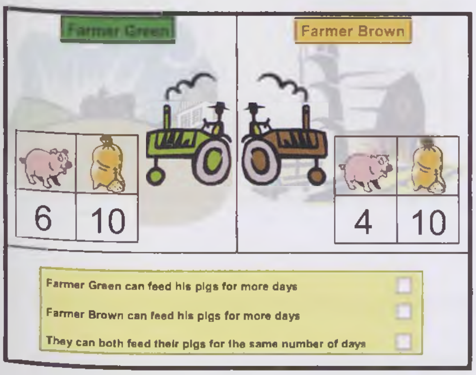
Will Farmer Blue be able to feed his sheep for more days?

Will Farmer Red be able to feed his sheep for more days?

Or will they both be able to feed their sheep for the same number of days?



Extensive inverse quantitative condition



Farmer Green has 6 pigs, Farmer Brown has 4 pigs.

Farmer Green has 10 bags of pig food; Farmer Brown has 10 bags of pig food.

Will Farmer Green be able to feed his pigs for more days?

Will Farmer Brown be able to feed his pigs for more days?

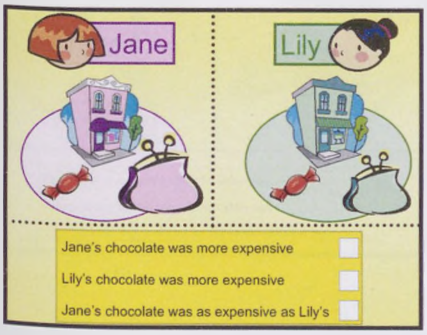
Or will they both be able to feed their pigs for the same number of days?

Figure 5.1 Examples of items from the quantitative condition

5.2.2.3 Relational Comparison Items

Children assigned to the relational comparison condition received parallel problems to those presented to the children in the quantitative condition but with relational rather than numerical information about the variables in the problem, as shown in Figure 5.2.

Intensive direct relational comparison item



Jane gets more pocket money than Lily does.

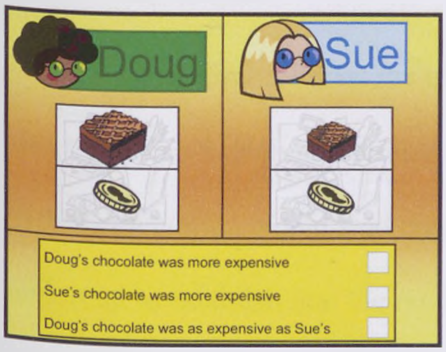
They both spend all their money on the same amount of chocolates.

Were Jane's chocolates more expensive?

Were Lily's chocolates more expensive?

Or were Jane's chocolates as expensive as Lily's?

Intensive inverse relational comparison item



Doug buys more chocolates than Sue does.

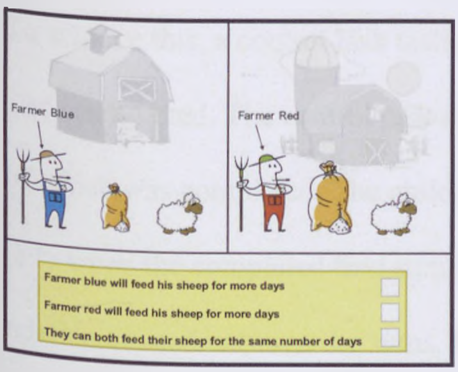
They both spent the same amount of money.

Were Doug's chocolates more expensive?

Were Sue's chocolates more expensive?

Or were Doug's chocolates as expensive as Sue's?

Extensive direct relational comparison item



Farmer Red has more sheep food than Farmer Blue.

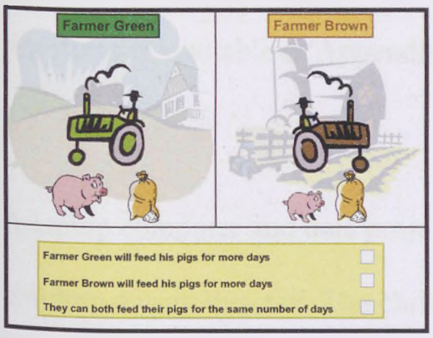
They both have the same number of sheep.

Will Farmer Blue be able to feed his sheep for more days?

Will Farmer Red be able to feed his sheep for more days?

Or will they both be able to feed their sheep for the same number of days?

Extensive inverse relational comparison item



Farmer Green has more pigs than Farmer Brown.

They both have the same amount of pig food.

Will Farmer Green be able to feed his pigs for more days?

Will Farmer Brown be able to feed his pigs for more days?

Or will they both be able to feed their pigs for the same number of days?

Figure 5.2 Examples of items from the relational comparison condition

5.2.2.4 Control Task

Study 3 was designed to test the hypothesis that numerical information could significantly affect children's relational thinking; the possibility was raised that the presence of numbers in the presentation would lead children to attempt computations. With this in mind it was important to establish that the random assignment of children to experimental conditions was successful and that both groups were comparable in terms of ability to understand number relations and perform computations.

To achieve this, a control task testing the children's understanding of numerical relations was constructed. The control task entailed presenting children with pairs of sums, the first of which was completed. The children then provided an answer to the second sum, but only when the completed first sum helped them answer the second one. Sometimes a clear relation existed between the sums, with the completed sum providing a strong hint to the answer of the second sum (e.g. 54+32=86, 55+32=?). In other pairs of sums there was either no clue in the first sum, or a potentially misleading clue in the first sum (e.g. 54+32=86, 540+32=?). The children were told that they would have a limited amount of time to work through the sums so they should skip those exercises where the first sum did not provide them with a clue. The children were given 20 minutes to work on the task; the total number of items was 75. The items covered number relations with addition, subtraction, multiplication and division problems. The items were presented in random order across a five-page booklet, starting with a simple addition problem. A full list of problems is available in Appendix 5.2.

5.2.3 Materials

A laptop computer and data projector displayed the problems in pictorial form to the children during the task. The children recorded their answers in booklets which contained the pictorial problems as displayed by the computer.

5.2.4 Ethical Approval

Ethical approval from the Social Sciences and Law Ethics Committee at Oxford Brookes University was obtained. An initial letter was sent to the head teachers of all the schools involved (Appendix 5.3). This letter contained a suggested letter of consent to be sent out to the parents of the children (Appendix 5.4). For each school the head teacher acted as the gatekeeper and was free to decide whether to seek permission from parents for their child's participation in the study.

5.2.5 Procedure

The children were randomly assigned to one of the two experimental groups, each consisting of roughly 10–15 children. The groups were taken to separate classrooms to complete the experimental task followed by the control task. The whole procedure took approximately 60 minutes to complete. Children received the experimental task before the control task so that there would be no contamination from the control to the experimental task.

5.2.5.1 Experimental Task

The procedure was identical for each group. The children were each given a booklet in which to record their answers. The researcher then read out the following instructions:

‘I am going to show you some pictures on the OHP/computer and for each picture I am going to read you a question, you should have in your booklet a picture that matches what I show you. You need to listen carefully and answer the question, when you have put your answer please turn over to the next page in your booklet and put your pencil down so I will see who is ready and who needs a bit more time.’

The researchers also answered any of the children’s queries before reading out the first item.

5.2.5.2 The Control Task

After the children had finished working on the main task, they received the control task booklet and the following instructions:

‘This task is about looking for clues that can help you solve sums. In this booklet are pages with lots of sums on them, you will see that the sums are together in pairs. The first sum in the pair already has the answer, but the second does not. What you need to do is look at the first sum with the answer and see whether it gives you a clue about how to solve the second sum, if the first sum doesn’t give you a clue then leave that question and look at the next pair. Because there are many sums here, you will not have enough time to answer everything, you should remember the game is about trying to find the questions with a clue that helps you. If you manage to find all the questions with clues if you still have time then you should go back and try to answer the ones without clues.’

After any questions from the children, the task started and the children worked individually under exam conditions.

5.3 Results

5.3.1 Overview

The results are presented in three sections. The first section considers the performance of children on the control task. Comparing performance between experimental conditions will provide important information for the analysis of main effects by showing whether the random assignment of children produced comparable groups. The second section will consider the main 'within participants' analyses to test the effects of type of quantity (Hypothesis 1) and relation (Hypothesis 2). The second section also considers the tests of 'between subjects' effects of experimental group. This section is important to the testing of Hypothesis 3 of the effect of qualitative and quantitative task presentations on performance. Section 2 also includes age comparisons to look more closely at the differences in performance on extensive and intensive quantities problems across the wider range of ages tested. The third section presents an exploratory analysis of the children's errors to consider how the presence of numerical information affected the children's judgements.

5.3.2 Children's Performance on the Control Task

The control task scores were analysed in order to confirm that the random assignment of the children between conditions had produced equivalent groups. Figure 5.3 shows a box-plot with the distribution of control task scores by group. An independent group t-test showed no significant group differences on control items ($t(242)=0.976$, $p=.330$). This means the groups were comparable in terms of general knowledge of numerical relations.

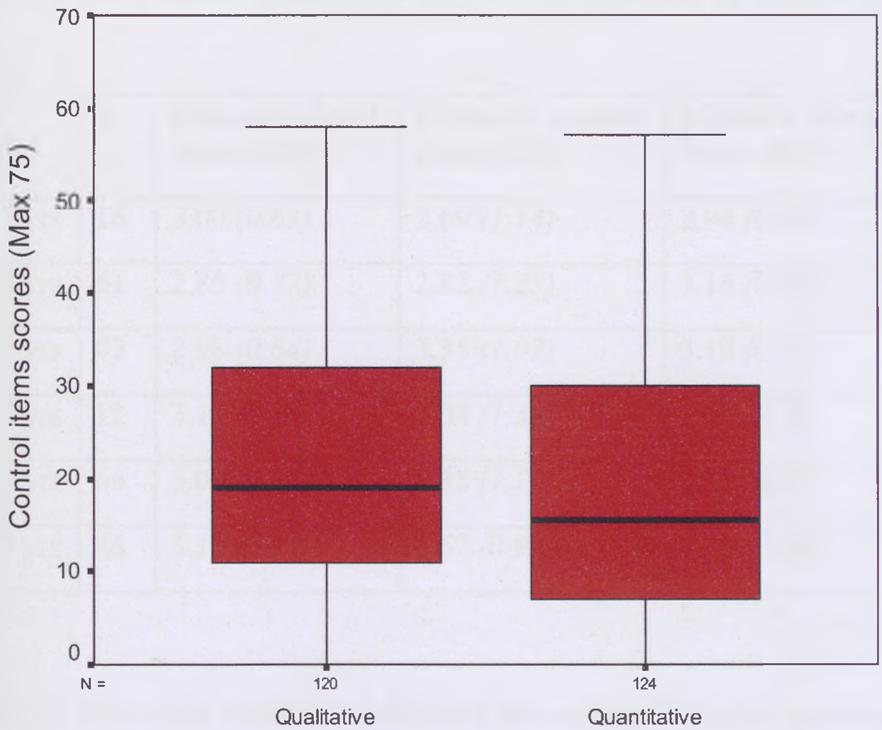


Figure 5.3 Box-plot for control items by type of information received

5.3.3 The Analysis of Main Experimental Effects

This section deals with the 'within participants' and 'between participants' analysis of the experimental task and directly tests hypotheses 1, 2 and 3. Children's performance on each item of the experimental task was coded as '1' for correct and '0' for incorrect, meaning that for each question type (e.g. extensive quantity direct relations problems) a maximum score of 4 was possible. Table 5.1 shows the mean scores and standard deviations by type of question, age level and type of information received.

Table 5.1 Mean Scores (Out of 4) and Standard Deviations by Type of Question

		N	Extensive direct Mean *(SD)*	Extensive inverse Mean *(SD)*	Intensive direct Mean *(SD)*	Intensive inverse Mean *(SD)*
Relational	7 yrs	16	3.00 *(0.63)*	2.69 *(1.14)*	2.94 *(0.93)*	2.19 *(0.98)*
	8 yrs	61	2.85 *(0.77)*	2.82 *(1.23)*	3.16 *(0.86)*	2.20 *(0.96)*
	9 yrs	43	2.98 *(0.64)*	3.35 *(1.02)*	3.19 *(0.82)*	2.67 *(1.19)*
Quantitative	7 yrs	12	3.17 *(0.58)*	2.08 *(1.38)*	3.00 *(0.74)*	1.67 *(1.23)*
	8 yrs	66	3.06 *(0.80)*	2.38 *(1.19)*	3.23 *(0.84)*	2.23 *(1.03)*
	9 yrs	46	3.15 *(0.82)*	2.57 *(0.98)*	3.26 *(0.80)*	2.63 *(1.10)*

Table 5.1 shows that in both the relational information and quantitative conditions children performed close to ceiling level on direct relations problems. For inverse relations problems there was more variation within the children's scores.

The analysis of distributions showed that in the relational information group the distribution of 9 year olds on extensive quantity direct relations problems was significantly negatively skewed (z=-2.31) meaning these problems were quite easy. It is recommended in Howell (1997) to use data transformation in order to reduce the problem of skewed distributions. As the problem in this case was negative skewness the data were transformed using an Arcsine transformation. This resulted in skewness scores in all distributions that were non-significant, therefore normal distributions could be assumed with this transformation for the purposes of data analysis.

Mauchly's diagnostic test for sphericity was shown to be non-significant which means that unadjusted F values could be reported. Levene's test of equal variances showed that equal variances could not be assumed as a significant result was found for direct relations extensive quantity problems (p<.0001). This means that the choice of post-hoc test must be able to cope with this. Field (2000) recommends employing Games-Howell post-hoc tests when equal variance cannot be assumed and when sample sizes differ between groups.

5.3.3.1 Within Group Comparisons

To test the experimental hypotheses 1 (main effect of quantity) and 2 (main effect of relation) relating to the replication of findings from Study 2, a repeated-measures ANOVA was employed with experimental group and age category as between participants measures; the dependent variables were children's scores on each repeated measure.

Significant main effects of type of quantity (extensive vs. intensive) were observed ($F_{1,238}$=2.86, *p*=.046) though the Cohen's d effect size was small (0.14 SD). There was also a significant main effect of type of relation (direct vs. inverse) ($F_{1,238}$=37.12, *p*<.0001) with a Cohen's d effect size of 0.73 SD which is considered a medium effect. In addition to significant main effects there was also a significant two-way interaction between quantity and relation ($F_{1,238}$=20.76, *p*<.0001); Figure 5.4 shows the nature of this interaction. The children's scores in the direct relations problems were at ceiling level for both extensive and intensive quantity problems. This means that it was not possible to make post-hoc comparisons between extensive and intensive direct relations problems as there was not enough potential for variation in the children's scores. Post-hoc comparisons of inverse relations problems were possible as the children's scores were not at ceiling level. For inverse relations problems, Bonferroni adjusted post-hoc tests showed that extensive quantity problems were significantly easier than intensive quantity problems (*p*<.05).

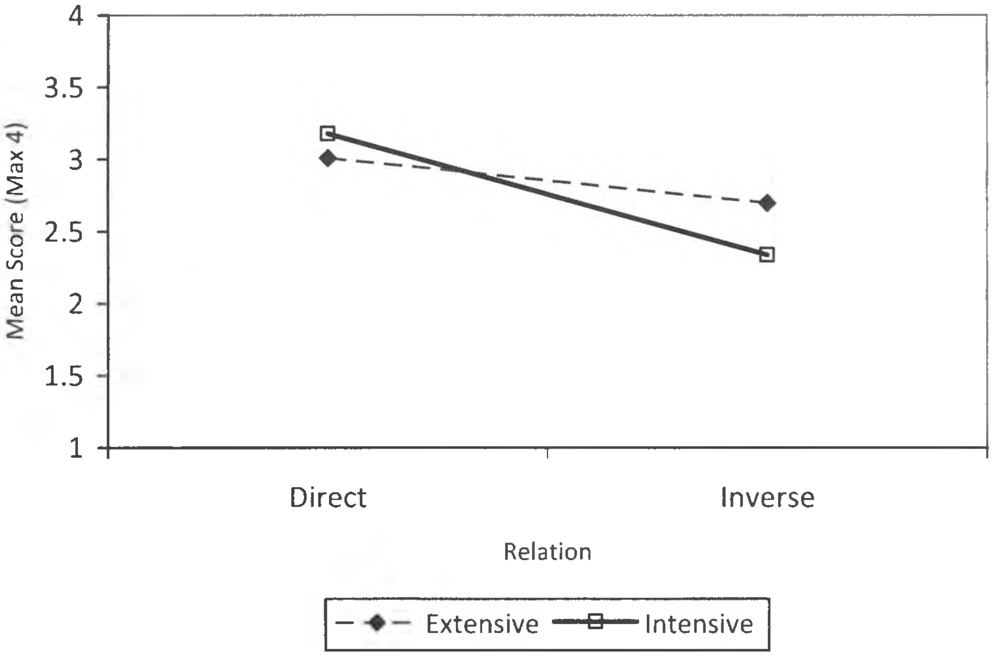


Figure 5.4 Quantity by relation interaction

The implications of this interaction for experimental hypotheses 1 and 2 are clear. Although there was a significant main effect of type of quantity the effect size was relatively small. The quantity by relation interaction showed that this small effect size was probably due to the ceiling level performance of children on direct relations problems. This means that because the direct relations problems were too easy for the children in Study 3, extensive and intensive quantity comparisons were only clearly interpretable on inverse relations problems. On inverse relations problems, extensive quantities were easier for children to solve than intensive quantity problems. As the children in Study 3 were only comparable on inverse relations problems the conclusion can be reached that in line with the first experimental hypothesis, extensive quantity problems were easier to solve than intensive quantity problems. For the test of the second hypothesis of a main effect of relation, the results clearly supported the notion that direct relations were significantly easier for children than inverse relations. This led to the acceptance of the second experimental hypothesis.

5.3.3.2 Between-Participants Effects

Study 3 had two between-participants factors. The first was the experimental manipulation of form of problem presentation (relational vs. quantitative), and the second was the age category of the children (7, 8, and 9 year olds). On the question of form of presentation there was no significant main effect ($F_{1,238}$=1.14, p=.144). The Cohen's d effect size was 0.1 SD which is considered a negligible effect size. There was a significant two-way interaction observed between form of presentation and relation ($F_{2,238}$=11.35, p=.001). Figure 5.5 shows the nature of the interaction. Games-Howell post-hoc tests showed that no differences existed between the groups' performances on direct relations problems while children in the relational information condition performed significantly better on inverse relations problems than children in the quantitative condition (p<.05). The Cohen's d effect size was 0.38 SD which is considered a small effect size. As children's performance in the quantitative condition group was not significantly better than that of children in the relational information group, the third experimental hypothesis was rejected. The presence of numbers did not facilitate children's relational thinking.

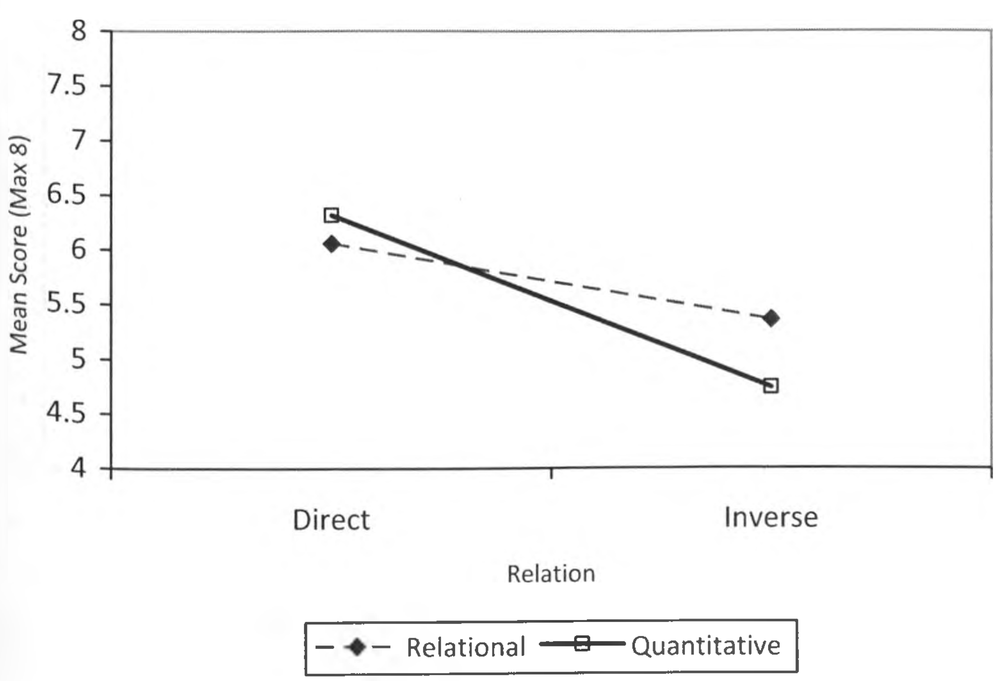


Figure 5.5 Form of presentation by relation interaction

On the question of how the wider range of ages than included in Study 2 influenced the results, significant between-participants age effects were found ($F_{2,238}$=5.94, p=.003). Games-Howell post-hoc tests showed that 9 year olds scored significantly higher than 7 year olds (p<.05) with a medium Cohen's d effect size 0.61 SD. The scores of 9 year olds were also significantly higher than 8 year olds (p<.05) with a medium Cohen's d effect size of 0.64 SD. The similar effect sizes observed in the comparisons of 7 and 8 year olds against 9 year olds indicates that children made little progress in their understanding of relational thinking problems between the ages of 7 and 8 years.

There was also a significant two-way interaction of relation by age ($F_{1,238}$=3.61, p=.029). Games-Howells post-hoc tests showed that the better performance of 9 year olds can be explained by an increase in performance on inverse relations problems over that of 7 (p<.05) and 8 year olds (p<.05). As children's performance at each age on direct relations problems remained flat, no significant between age differences on direct relations problems were observed (see Figure 5.6).

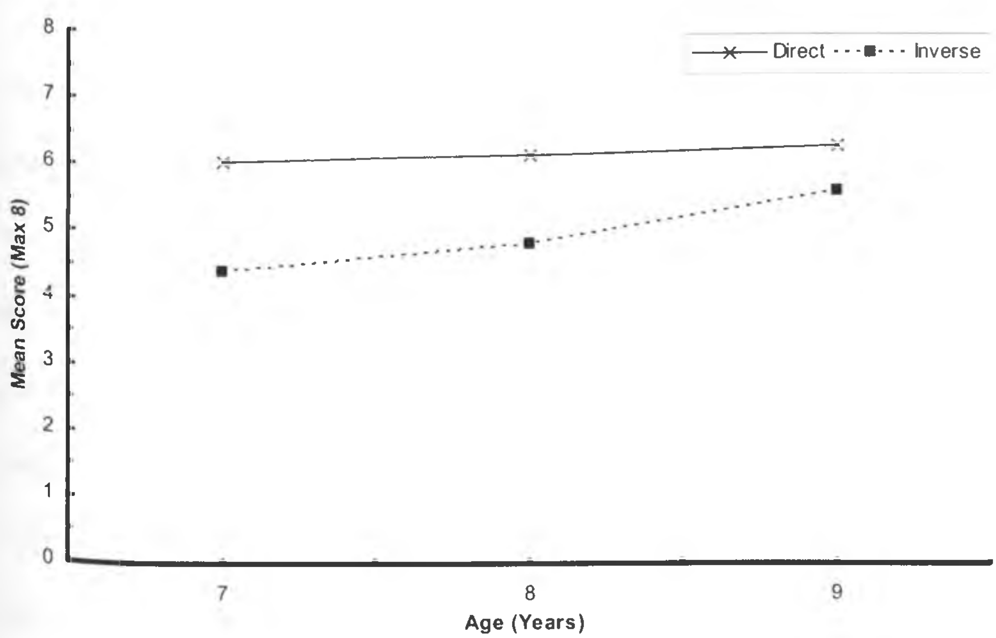


Figure 5.6 Relation by age interaction

5.3.4 Exploratory Analysis of Inverse Relations Problems

The results of the between-subjects analysis showed that when considering the effect of the form of problem presentation, the use of numbers did not facilitate children's performance on relational thinking problems. The post-hoc analysis showed that on inverse relations problems, those presented using relational information actually had a small positive effect on the children's performance. The reasons behind this were explored by looking at children's judgements on inverse relations problems.

The children were solving multiple-choice problems, for which three possible answers were available, so two types of error were possible. The first type of error would be to attend to only one variable across both direct and inverse problems, For example, in cost problems, reasoning that more money means the product is more expensive would lead to the correct answer for direct relations problems. However, on the inverse item, thinking only about money would lead children to reason that the same amount of money would mean the same cost, regardless of the size of the product purchased. The second type of error would be to use 'the more A, the more B' rule.

To understand why inverse relations problems were easier to solve when presented with relational information, children's responses across all inverse problems were considered. For the observed pattern of results there are two possibilities. The first is that the presence of numerical information resulted in a general increase in problem difficulty. If this is the case then the same pattern of errors should occur between relational and quantitative conditions. The second possibility is that the presence of numerical information led children to make more errors of a particular type. In this case it would be expected that the pattern of errors would differ between children in the relational and quantitative groups. Figure 5.7 shows the proportion of error rates by group. The proportions do not add up to 1 as children also produced correct answers.

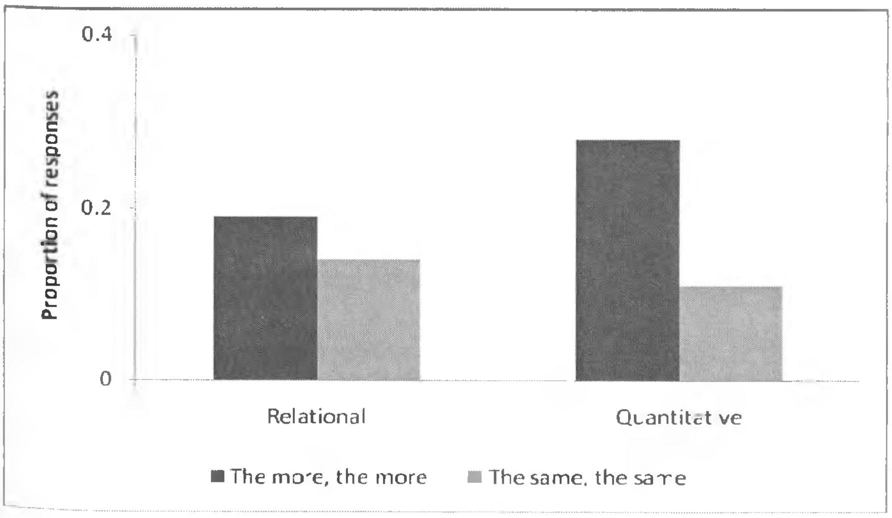


Figure 5.7 Proportions of erroneous judgements on inverse relations items

Figure 5.7 shows that the second interpretion is more accurate as children in the quantitative condition made a much greater proportion of 'the more, the more' errors compared to 'the same, the same' errors. In the relational information condition the proportions of error types were quite similar. It is possible that the numerical information was important in indicating to the children that they should consider both quantities, but that a lack of experience in working with inverse numerical relations made it difficult for children to apply the inverse relation.

5.4 Discussion

5.4.1 The Distinction Between Extensive and Intensive Quantities

The first aim of Study 3 was to replicate the findings of Study 2 which had shown that extensive quantity problems were easier than intensive quantity problems (Hypothesis 1), and that direct relations problems were significantly easier than inverse relations problems (Hypothesis 2). Although there was a clear main effect for type of relation with direct relations problems significantly easier leading to the acceptance of the second hypothesis, tests for main effects of quantity (Hypothesis 1) were more difficult to interpret. Although there was a significant main effect of quantity, the effect size was small and there also appeared a significant quantity by relation interaction. Exploration of the data showed that the children in Study 3 were performing close to ceiling level on direct relations problems. This meant that these problems were too easy to allow a comparison of extensive and intensive quantities problems. On inverse relations problems, those involving extensive quantities were significantly easier for children than those involving intensive quantities.

Age differences between the children participating in Study 3 and those who participated in Study 2 provides a possible explanation for the ceiling-level performance reported on direct relations problems in Study 3. In Study 2, the children were between 7 and 8 years of age with the majority of children falling into the younger category. In Study 3, the children were drawn from three age categories (7, 8, and 9 years) with the majority of children falling in the two latter categories. It is a well-established finding in both the work on intensive quantities (Piaget, 1970; Piaget and Inhelder, 1975; Strauss and Stavy, 1982), and more recent work on sharing (Correa, Nunes, and Bryant, 1998; Squire and Bryant, 2003) that the period around the age of 7 to 8 years is crucial in the development of children's understanding of inverse relations. Indeed the findings reported in Study 1 reiterated this finding. Study 2 showed that at this crucial period of in the development of children's understanding of inverse relations, the distinction between extensive and intensive quantities is important. For the older sample participating in Study 3, the results showed that the general distinction between extensive and intensive quantities is less important. This is primarily because direct relations problems are now too easy for these children. These children were not entirely comfortable with the more difficult inverse relations, and on this type of problem performance on those involving intensive quantities lagged behind that on comparable problems involving extensive quantities. This finding adds to the general findings in the literature which report the distinction between direct and inverse relations problems as significant (Suber, 1980; Acredolo, Adams and Schmid, 1984; Squire and Bryant, 2003). The current findings also suggest that the acquisition of the inverse relation is not an all or nothing process, as shown by the fact that the participants found inverse relations with intensive quantities more difficult than the equivalent problems involving extensive quantities. This occurred even though the number of quantities under consideration in the problems remained the same.

5.4.2 Form of Presentation

The second aim of Study 3 was to test the extent to which the form of problem presentation would affect children's reasoning. It had been shown in the literature on mixing and sampling intensive quantities that the use of numerical information in presentation led children to try and carry out computations (Cowan and Sutcliffe, 1991; Harel, Behr, Lesh and Post, 1994; Desli, 1999). The impact of this distinction on direct and inverse relations problems was untested.

Although the tests of main effects on this question were not significant (as performance on direct relations problems was good for both groups) an interaction between presentation and relation showed children's performance on inverse items to be significantly higher in the relational information condition. This lead to the acceptance of the third experimental hypothesis. The exploratory analysis of children's errors showed that children made a greater proportion of 'the more, the more' errors in the quantitative information condition.

This finding adds to those of Harel et al. (1994) and Schwartz and Moore (1998) with sampling problems showing that numerical information has a negative effect on children's performance. This is because the numbers led children to make less sophisticated additive comparisons when computations were not needed. The current finding is also consistent with the joining of intensive quantity studies by Cowan and Sutcliffe (1991) and Desli (1999), which showed that the presence of numbers does not provide children with clues about the need to use averaging, so children use the available numerical information in a less sophisticated adding strategy. The results of Study 3 fit with this pattern of numerical information leading to naive additive comparisons (Resnick and Singer, 1993). However unlike in sampling and joining of intensive quantity problems, the presence of numerical information has the most potential to help children make the logical connection between quantity and direction of relation themselves. Children who do not have a systematic understanding of inverse relations and who are left to make the connection by themselves appear to favour a familiar 'more A, more B' interpretation of the numbers rather than a 'same A, same B' interpretation. It appears in accordance with intuitive rules theory (Stavy and Tirosh, 2000) that the presence of numbers is more likely to lead children to use intuitive knowledge of number relations rather than think about the logic of the problem.

As the studies conducted so far have all employed a comparison of relations methodology, it is important to consider how the distinction between extensive and intensive quantities and the difficulty of inverse reasoning extends to problems that require computations. This question forms the basis of Study 4.

CHAPTER 6

STUDY 4 COMPARISON OF EXTENSIVE AND INTENSIVE QUANTITY PROBLEMS IN COMPUTATIONAL CONTEXTS

6.0 Aim

The aim of Study 4 is to compare children's performance on extensive and intensive quantity problems involving direct and inverse relations when calculations are required. This extends the analysis of studies 2 and 3 that only considered children's performance on non-computational problems.

6.1 Introduction

In the first three studies of this thesis it was shown that between the ages of 7 and 9 years children begin to solve non-computational inverse relations problems systematically. It was also shown in studies 2 and 3 that children solved non-computational problems earlier in extensive than in intensive quantity settings. The final study of this thesis focuses on how children handle the distinction between extensive and intensive quantities when problems require computations.

The quantities which comprise the extensive and intensive quantity problems tested so far are multiplicatively related. When presented in their simplest form as the non-computational problems in studies 2 and 3, children could reach the solution without needing to consider the quantities multiplicatively. Designing a study to test children's computational success with relational thinking problems will provide a direct test of how children cope with the additional requirement of interpreting the scalar relations as multiplicative.

6.1.1 Assessing Children's Computational Skill via Missing Value Proportion Problems

Studies of children's computational skills with proportional relations has been achieved primarily in the literature with the use of missing value proportions problems which Reis, Behr, Lesh, and Post (1985) define as:

'A missing data proportion problem then begins with two conditions: that (a) the multiplicative relationship (i.e., the ratio) is constant and (b) a given transformation on one of the two components is given. The problem solver must then determine (a) the transformation to perform on the other component and (b) the result of this transformation. That is, the problem solver must determine the required action under the given conditions.' (p.354)

Missing value proportions problems in proportional reasoning research have been used to document children's development from an initial additive conception of the relation between quantities to a multiplicative conception. This discussion of children's transition from additive to multiplicative forms of thinking has long represented a cornerstone of proportional reasoning research (Piaget, Grize, Szeminska, and Bang, 1977; Karplus, Pulos, and Stage, 1983; Hart, 1984). The literature also shows that this transition from additive to multiplicative reasoning is a gradual process. Even when children are used to working with relations multiplicatively there is a tendency to revert to additive strategies when problem difficulty increases. For example Karplus, Pulos, and Stage (1983) looked at the solutions reached by 11–14 year olds in comparison of lemon/sugar ratio problems. The children were given both equivalent and non-equivalent ratio comparison problems. If the children made a judgement of non-equivalence they were asked to describe what adjustments they would make to the drinks so they would have the same taste. Karplus, Pulos and Stage report that, as the ratios which the children were asked to compare became more difficult (i.e. the numbers in the problem were not easily divisible), children were less likely to offer multiplicative-based solutions, and were instead more likely to offer additive solutions. Others such as Kurth (1988) and Zealzo and Shultz (1989) have shown that children have difficulty working with inverse computational problems, though the children in these studies only worked with extensive quantity problems. Following work with 11 year olds on a balance beam task, Ferrandez-Reinisch (1985) proposed that two stages of development can be seen when children are asked to solve missing value problems. At the first stage children understand the direction of the relation, but have an inability to compute the answer. At the second stage children are able to coordinate the numerical relations and solve the problem.

For the extensive and intensive quantity problems studied so far a degree of caution is required when interpreting how findings from the general proportional reasoning literature reflect children's computational success on direct and inverse relations problems. The need for caution stems from the prevalence of missing value proportional reasoning problems reported in the literature which is based on the isomorphism of measures structure (Vergnaud, 1983). There is a perfectly good reason why children's understanding of proportional reasoning is assessed with the isomorphism of measures structure as it represents the most basic proportional reasoning computational problem (direct relation between two quantities). For the purpose of the current thesis there are three related reasons why problems following the isomorphism of measures structure are not suitable for assessing children's computational success with relational thinking problems. These can be illustrated by using Vergnaud's (1983) example of an isomorphism of measures problem:

"If four cakes are priced at five pence each, how much does the person have to pay?" (p.129)

Firstly, children only have to think about a direct relation, in this case between cakes and money, as the third quantity in the problem (in this case the intensive quantity of relative cost) is both a constant and implicit in the problem structure. This leads to the second reason: due to the isomorphism of measures structure keeping the third and usually intensive quantity implicit, missing value problems measure children's success at maintaining proportional equivalence. Proportional equivalence in intensive quantity settings is known to be a difficult logical step for children and one which follows an awareness of direct and inverse relations (Noelting, 1980; Strauss and Stavy, 1982). This leads to the third reason: when a problem involves maintaining a proportionally equivalent relation, the issue is how children discover the scalar relation. In Vergnaud's problem it is the understanding that replicating the measure of cakes from 1 to 4 requires a replication of the number of 5p from 1 to 4 (or 5p to 20p). For the relational thinking problems currently under investigation the link between the direct and inverse relations in studies 1 to 3 and computational success is not about the discovery of the scalar relation but about interpreting a known scalar.

To explore children's computational skill on direct and inverse relations problems, Study 4 adapts a method of missing value problem construction reported in Piaget, Grize, Szeminska, and Bang's (1977) feeding of eels study. Here Piaget et al were interested in the acquisition proportional relations in children when direct relations occurred between two dimensions (eel size and quantity of food). Instead of giving children a standard missing value problem in which the multiplicative relation between quantities was implicit, Piaget et al presented children with the scalar relation between food and eels size and reported the children's interpretations of this relation. For example the children were told that eel B eats twice as much food as eel A and then asked if eel A is given two food pellets how many food pellets should be fed to eel B? Under these conditions Piaget et al showed that by around 10 years of age, children could establish accurate multiplicative relations between quantities. However, Piaget et al's study considered only direct relations between two quantities (isomorphism of measures structure) so the question of inverse relations was never raised. Adopting Piaget et al's (1977) methodology makes it possible to create direct and inverse computational problems involving intensive and extensive quantities. In Study 4 children will be provided with numerical information about two of three quantities following the common missing value problem structure. In addition to this, children will receive explicit information about the scalar relation of the third (intensive or extensive) quantity before being asked to calculate the value of the missing quantity. For example, an inverse intensive quantity problem in this style might be: ‘Drink A is made with 3 measures of orange juice mixed with 6 measures of water. Drink B is made using 3 measures of orange juice. How many measures of water need to be mixed with Drink B so it will taste twice as orangey as Drink A?’ This line of questioning will allow the examination of children’s performance on direct and inverse computational problems, independently from the requirement to think about proportional equivalence.

6.1.2 Children’s Performance on Extensive and Intensive Quantity Computational Problems in the Literature

Currently within the literature, no systematic comparisons exist between extensive and intensive quantity computational problems. There is evidence in the literature to suggest that presenting problems with extensive or intensive quantities leads children to different solution strategies.

The influence of a third extensive quantity on children’s reasoning strategies was commented on by Kaput and West (1994) during their post-hoc examination of the solutions given by children aged 11–14 years to missing value proportional reasoning problems following the pre-test of a classroom-based teaching experiment. Kaput and West were interested in how children’s informal reasoning strategies are influenced by various factors i.e. numerical arrangement or problem context. When discussing the semantic features of missing value problems, Kaput and West highlighted how one problem known as the ‘placemats’ problem was solved more easily than a ‘salad dressing’ problem. The placemats problem was a missing value proportional reasoning problem involving relations between three extensive quantity problems (silverware, china, placemats):

'A large restaurant sets tables by putting 7 pieces of silverware and 4 pieces of china on each placemat. If it used 392 pieces of silverware in its table settings last night, how many pieces of china did it use?' (Kaput and West, 1994, p.267)

The salad dressing problem was also a missing value proportional reasoning problem involving the relation between two extensive quantities (oil and vinegar) and a third intensive quantity (taste):

'To make Italian dressing, you need 4 parts vinegar for 9 parts oil. How much oil do you need for 828 ounces of vinegar?' (Kaput and West, 1994, p.268)

Although both problems were difficult, the children were more successful on the placemats problem (32% pass rate for the placemats problem, 13% pass rate for salad dressing problem). Kaput and West (1994) reported that the placemats problem was easier as the third quantity in the problem (placemats), acted as a semantic holder upon which children could imagine quantities of silverware and china being placed. This facilitated a 'building-up' strategy (Hart, 1984) with which children were successful. For the salad dressing problem Kaput and West suggested that because the problem involved mixing ingredients together, the quantities of oil and water lose their separate identities. This contrasts with the placemats problem where the quantity placemats held the quantities of silverware and china together while preserving their separate identities. Kaput and West suggested that the salad dressing problem could be made easier for children if the quantity 'bottles of salad dressing' was added. This would change the problem from one of intensive quantities to one involving three extensive quantities (oil, vinegar, bottles of salad dressing).

It must be made clear that Kaput and West's (1994) work was exploratory, and wasn't designed to provide a systematic test of the distinction between extensive and intensive quantity problems. Kaput and West's problems were also traditional missing value problems concerning the maintenance of proportional equivalence and not direct and inverse relations. However, the author's post-hoc analysis of missing value problems does raise the possibility that with computational problems children may deal with intensive and extensive quantity computational problems in different ways.

6.1.3 Rationale

No systematic comparisons of extensive and intensive quantities relational thinking computational problems exist in the literature. It is possible to test these problems by adapting a methodology originally proposed by Piaget et al (1977). There is also some anecdotal evidence in the literature provided by Kaput and West (1994) with proportional equivalence problems which suggests that problems involving extensive quantities facilitate children's informal computational strategies. To look at this question systematically study 4 has two experimental hypotheses relating to the test of main effects for relation (hypothesis 1) and a main effect for type of quantity (hypothesis 2).

6.1.4 Hypotheses

The following experimental hypotheses were generated to test differences between direction of relation (Hypothesis 1) and quantity type (Hypothesis 2).

Hypothesis 1:

'Direct relations items will be significantly easier than inverse relations problems'

Hypothesis 2:

'Extensive quantity problems will be significantly easier than intensive quantity items'

6.2 Method

6.2.1 Participants

A total of 121 children from years 3 and 4 (47 boys and 74 girls) participated in the study from five Oxfordshire primary schools and one London primary school. The mean age of the children was 8:9 years with a standard deviation of 6.7 months with a range of 7:7 to 9:9 years. The children participating in the current study were drawn from the same cohort and tested alongside the children who participated in Study 3.

6.2.2 Design

Study 4 employs a mixed model 2x2x3 design: quantity type (extensive vs. intensive – within participants) by relation type (direct vs. inverse – within participants) by age group (7 years vs. 8 years vs. 9 years – between participants).

6.2.2.1 Experimental Task

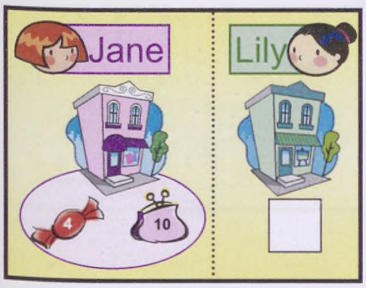
Children solved missing value comparison problems with extensive or intensive quantities. In each case, children were asked to find the value of a missing extensive quantity which was either directly or inversely related to the intensive or third extensive quantity in the problem. The items were designed to be within the computational limits of the youngest children participating (Year 3). For this reason the problems all involved multiples of 2, 3, and 4 in accordance with The National Numeracy Strategy (DfEE, 1999) which states that Year 3 children should:

'know by heart 2, 5, and 10 times tables (and) begin to know 3 and 4 times tables' (p.14)

The children were all tested towards the end of the summer term so it was expected that these multiples had been covered in their classes.

The problems all followed the same format: comparisons were made between two characters on the basis of a particular intensive or extensive quantity. It is possible to illustrate this using the first problem highlighted in Figure 6.1 as a template. Numerical information was always provided for two of the extensive quantities concerning one of the characters (the number of chocolates Jane bought and the price she paid). Information was then given about the scalar relation on a third quantity (extensive or intensive) between the characters (Jane and Lily). In the cost example below, this involved telling children that the second character (Lily) went to a shop were the chocolate was twice as expensive. Information was then given about the quantity which was controlled between characters (Lily bought the same amount of chocolate as Jane). The children were then asked to use this information to compute the value of the missing quantity (How much money will Lily pay?). Figure 6.1 shows examples of how the question format was used in extensive and intensive quantity settings. A full list of the problems can be found in Appendix 6.1.

Intensive direct

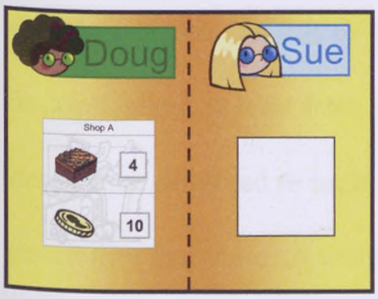


Jane buys 4 chocolates for 10p.

Lily goes to a shop where chocolate is twice as expensive and buys the same amount of chocolate as Jane.

How much money will Lily pay?

Intensive inverse

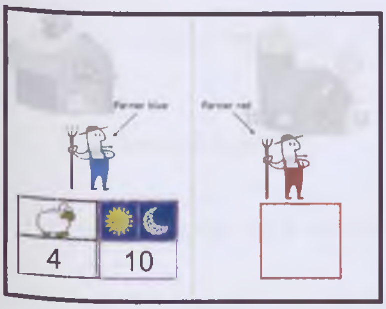


Doug buys 4 chocolates for 10p.

Sue goes to a shop where chocolate is twice as expensive and spends the same amount of money as Doug.

How many chocolates will Sue get?

Extensive direct

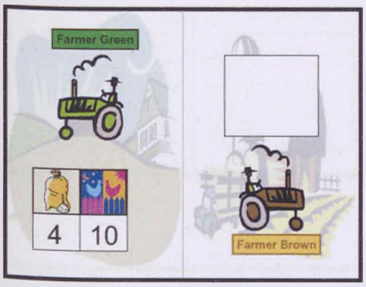


Farmer Blue has 4 sheep. He bought enough food to feed them for 10 days.

Farmer Red bought twice as much food and has the same number of sheep.

How many days will his food last?

Extensive inverse



Farmer Green has 4 bags of pig food. It will feed his pigs for 10 days.

Farmer Brown has twice as many pigs and the same amount of food.

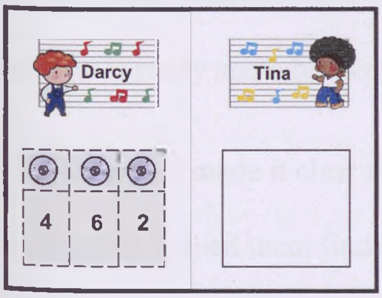
How many days will his food last?

Figure 6.1 Examples of computational items

As the problems were designed to present children with the scalar relation ‘x times’, there was the concern that this term may be seen as synonymous with the operation of multiplication, at least from the children’s perspective (Nesher, 1988). To address this the items were designed to include two relational terms ‘x times’ and ‘same’ to reduce the possibility that children would just blindly multiply without considering the quantities in the problem. It was hoped that using two relational terms within the problem would encourage children to consider the relations between quantities more carefully rather than just multiply them. To test the extent to which relational language might lead children to blindly multiply numbers, a set of control items were also designed.

6.2.2.2 Control Items

The control items took the form of additive reasoning items designed to contain the same relational language terms as the experimental items (x times, same). It is expected that if ‘x times’ is a blind cue to multiply, then children should make multiplication errors on additive control items. A total of four control items was included at random points in both conditions, see Figure 6.2 for an example.



Darcy buys CDs 3 times. On the first day he buys 4 CDs, on the second 6 CDs, and on the third 2 CDs. Tina buys the same amount of CDs as Darcy but all in one day. How many CDs did Tina buy?

Figure 6.2 Example of a control item

6.2.3 Materials

A laptop computer and data projector displayed problems in pictorial form to the children during the task. The children recorded their answers in booklets containing the pictorial problems as displayed by the computer.

6.2.4 Ethical Approval

Ethical approval from the Social Sciences and Law Ethics Committee at Oxford Brookes University was obtained. An initial letter was sent to the head teachers of all schools involved (Appendix 5.1). This letter contained a suggested letter of consent to be sent out to parents of the children (Appendix 5.2). For each school the head teacher acted as the gatekeeper and was free to decide whether to seek permission from parents for their child's participation in the study.

6.2.5 Procedure

6.2.5.1 Experimental Task

The children were given a booklet in which to record their answers. The researcher then read out following instructions:

'I am going to show you some pictures on the OHP/computer and for each picture I am going to read you a question, you should have in your booklet a picture that matches what I show you. You need to listen carefully and answer the question, when you have put your answer please turn over to the next page in your booklet and put your pencil down so I will see who is ready and who needs a bit more time.'

The researcher made it clear to the children that they could write or draw anything on the booklet that helped them find the answer. The researcher also answered the children's queries before reading out the first item.

6.3 Results

The results are presented in three parts. The first part of the results reports the analysis of the control items. The analysis of control items will help to determine whether the language used for experimental problems led children to blindly multiply numbers whenever they heard the expression ‘x times’. The second part of the results reports the test of main effects. This section covers the test of children’s performance on direct vs. inverse problems relevant to Hypothesis 1 and on extensive vs. intensive quantity problems relevant to Hypothesis 2. The third part of the results reports on an exploratory analysis of children’s responses for each problem type. The children’s performance in Study 4 is contrasted in this third part with the performance of children who participated in Study 3. The comparison of these groups provides a tentative interpretation of the findings reported in the analysis of main effects.

6.3.1 Control Items

The aim of the analysis of control items is to determine whether the term ‘x times’ led children to blindly multiply even on items which required solution by addition. If this were the case, one would expect low scores on this task, and that a large number of errors would have resulted from multiplying a number in the problem by ‘x’. One would not expect a negatively skewed distribution, approaching ceiling. In contrast, if the children are not biased by the expression ‘x times’, the distributions should be negatively skewed because children at this age level are expected to be able to add without difficulty.

The distribution of children’s scores in figure 6.2 shows that the children performed very well on the control problems, with the majority of children solving all four control problems correctly.

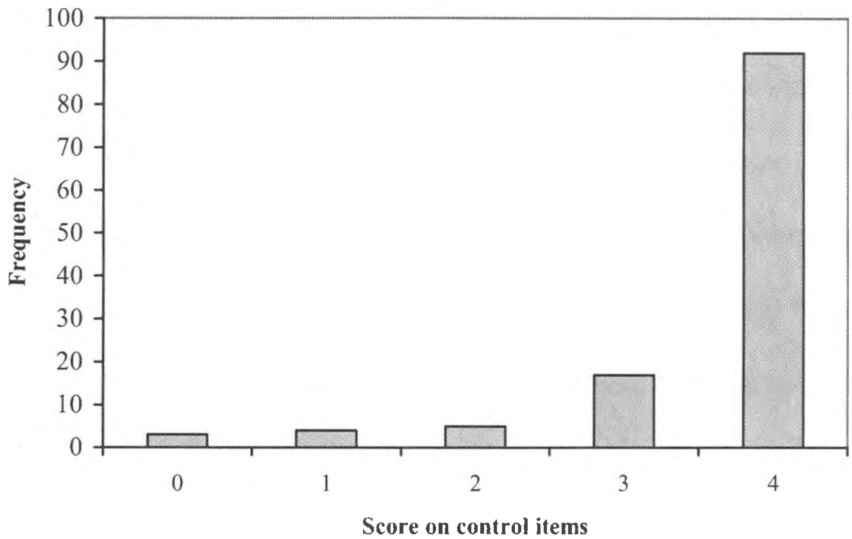


Figure 6.2 The distribution of children's scores on control items

The extremely high levels of perfect performance on the control items led to significant negative skew (z=-11.33). It can be concluded that, as children performed well on control items, the term 'x times' was not used as a cue to blindly multiply numbers.

6.3.2 Comparison of Extensive and Intensive Quantity Problems Between Conditions

The analysis of main effects provides a direct test of the two hypotheses generated for this experiment. The children's responses on each item of the experimental task was coded as 1 for correct and 0 for incorrect, this means for each question type (e.g. extensive quantity direct relations problems) a maximum score of 4 was possible.

Table 6.2 Mean Scores and Standard Deviations by Age and Type of Question (Maximum Score 4)

Age (years)	N	Extensive direct Mean *(SD)*	Extensive inverse Mean *(SD)*	Intensive direct Mean *(SD)*	Intensive inverse Mean *(SD)*
7	15	1.30 *(1.29)*	0.67 *(0.90)*	1.40 *(1.24)*	0.53 *(1.06)*
8	61	1.97 *(1.15)*	1.25 *(1.22)*	1.93 *(1.33)*	1.44 *(1.25)*
9	45	2.04 (1.21)	1.40 (1.19)	2.27 (1.36)	1.42 (1.23)

Table 6.2 shows that the mean scores are low at each age level and for each problem type. This suggests the additional requirement of interpreting a scalar relation made the problems more difficult to solve. Even the oldest age group (9 years) had only a mean success rate of 50% on direct relations problems. It is also clear from the children's mean scores that the distinction between direct and inverse relations provides a bigger contrast in performance for this sample than the distinction between extensive or intensive quantities.

For the test of main effects, Mauchly's Test of Sphericity was shown to be non-significant which means that unadjusted F scores will be reported. Levene's test of equal variances was also not significant meaning that equal variances could be assumed. The analysis of main effects was conducted using repeated measures ANOVA with age category as a between-participants factor; the dependent variables were the children's total scores in each problem category. There was a significant main effect of relation ($F_{1,118}$=48.189, p<.0001) with a medium Cohen's d effect size of 0.6 SD. There was no significant main effect of quantity ($F_{1,118}$=0.542, p=.463). The Cohen's d effect size was 0.07 SD which is considered an extremely small effect. This indicates that for the age groups tested the distinction between extensive and intensive quantity problems was less important than the calculation requirement which made all the problems difficult.

Between participant effects of age ($F_{2,118}$=3.552, p=.032) were also observed. For post-hoc testing it is recommended in Field (2000) that when equal variance can be assumed but sample sizes vary greatly, Hochberg post-hoc tests should be employed. Hochberg post-hoc tests showed that 7 year olds scored significantly lower than 9 year olds (p<.05) with a large Cohen's d effect size of 0.85 SD. There were no significant interactions. This indicates that general awareness of multiplicative relations on both direct and inverse relations is much greater in 9 year olds than in 7 year olds. In terms of the two experimental hypotheses, the first experimental hypothesis can be accepted, as direct relations problems were shown to be significantly easier than inverse relations items. It not was possible to accept the second experimental hypothesis, as the additional difficulty of performing calculations greatly increased the difficulty of all problem types for children in the age groups tested.

6.3.4 Exploratory Analysis of Children's Responses

The aim of this exploratory analysis of children's responses is to identify the reasons why the children's performance in Study 4 was poorer than the performance of the children who participated in Study 3. It is possible to consider the two studies together, as both samples were drawn from the same cohort. The items given to children in Study 4 were also designed specifically as computational versions of the non-computational problems given to children during Study 3.

Two contrasting findings between Study 4 and Study 3 are considered with this exploratory analysis. The first finding is why the ceiling level effect on direct relations problems observed in Study 3 was matched by poor scores on direct relations problems in the current study. The children's poor performance on direct items is particularly puzzling because the children received explicit information about the scalar relation in multiples with which the children were expected to be familiar. For example, if Mary's chocolate is twice as expensive as Paul's, it was expected that it would be obvious to the children that they should multiply the amount that Paul pays by 2 in order to find out how much money Mary paid. The second contrast in results between Study 4 and Study 3 is the distinction between children's performance on inverse relations problems. In Study 3 inverse relations problems were easier in extensive quantity contexts, while in Study 4 no significant differences were observed.

The comparison of children's performance between studies 3 and 4 will allow the consideration of two possibilities for these contrasts, firstly that the children's performance with computational problems was generally suppressed because the children found multiplication and division difficult and they miscalculated. The second possibility is that children's responses in Study 4 reflect the additional difficulty of needing to interpret the scalar relation in the computational problems as multiplicative, something not required to solve the non-computational problems of Study 3.

6.3.4.1 Coding the Responses of Children on Computational Problems

To look at the contrasting results between Study 4 and Study 3, children's responses in Study 4 were coded by type of calculation performed. This method of coding leads to three distinct categories. The first category includes only correct responses. The second category includes incorrect responses that can be traced to the child performing the wrong type of calculation. As the data is taken from a whole class task rather than from interview data, there is a third category consisting of incorrect responses where the reasoning behind the response has an unclear relation to the numbers presented in the problem. Although the proportion of correct and unclear responses are reported. it is the second category of responses (those that can be traced to the use of incorrect calculations) that has the potential to provide the most interesting insights into the contrast between children's performances in Study 4 and Study 3.

Focusing primarily on the children's responses that can be traced to performing the wrong calculations, a further two types of error are possible. The first type of error occurred when children interpreted the relations between the quantities in the problem as additive rather than multiplicative. As mentioned previously, this type of error is well reported within the wider proportional reasoning literature (Piaget et al, 1977; Karplus, Pulos and Stage, 1983; Hart, 1984). As the problems presented in the current study are also about the direction of relation and not maintaining proportional equivalence, a second type of computational error could occur if children incorrectly assumed the direction of quantity change. This would lead them to produce an answer consistent with a direct relations interpretation of an inversely related quantities problem (or vice versa). Children's errors with computational problems can result from either or both of these types of misconception. To clarify how these two types of error might interact, Table 6.4 shows the possible calculations children might attempt when making different assumptions about the problem structure and direction.

Table 6.4 Possible Calculations Carried Out on Depending on Children's Knowledge of the Problem Structure and Direction of Relation

Does the child correctly assume the problem structure?	
Yes Multiplicative structure	No Additive structure
Direct relation/ **Multiplication** Inverse relation/ **Division**	Direct relation/ **Addition** Inverse relation/ **Subtraction**

The task was delivered during a whole class session, and although encouraged to show their working out, the children did not do this consistently. To analyze children's responses, inferences have to be made about the type of calculation children used to arrive at an answer. For this reason this analysis should be treated as exploratory. To illustrate how children's responses were coded, the following example of an inverse relations taste problem will be used: 'Zico's drink had 8 lumps of sugar mixed with 12 measures of water. Joe's drink has the same amount of sugar but tastes 4 times sweeter. How many measures of water are in Joe's drink?' In this example a child's response would be coded as an inappropriate use of multiplication if the child had produced the answer 48 (12×4). An addition error would be coded if the child produced the answers 16 (12+4) or 20 (8+12). Subtraction errors were coded for the responses 4 (12-8). If the child produced the answer 8 which can result from the subtraction (12-4) or by simply repeating the number of sugar lumps in the problem, then a code of subtraction error was only applied if the child's protocol contained evidence of subtraction. If the origin of the answer could not be determined, the unclear code was applied. The unclear code was also used when children's responses had no clear connection to the numbers in the problem (e.g. 11).

6.3.4.2 Children's Responses by Problem Type

The children's responses are reported using proportions and not frequency of response. The reason for this is that one intensive quantity problem involving cost was removed from this analysis as the correct answer could be achieved through either dividing by or subtracting 2 from the target quantity (see Figure 6.1). Table 6.5 therefore shows the proportion of responses by item type.

Table 6.5 Proportion of Strategies by Item Type

Extensive direct	**Extensive inverse**	**Intensive direct**	**Intensive inverse**
.48 (correct)	.31 (correct)	.50 (correct)	.28 (correct)
.17 (multiplication using the wrong numbers)	.17 (multiplication)	.14 (multiplication using the wrong numbers)	.29 (multiplication)
.00 (division)	.04 (division)	.02 (division)	.03 (division)
.14 (addition)	.12 (addition)	.19 (addition)	.17 (addition)
.03 (subtraction)	.21 (subtraction)	.01 (subtraction)	.13 (subtraction)
.18 (unclear)	.19 (unclear)	.14 (unclear)	.10 (unclear)

The main point of interest from this general summary of children's responses is that from the majority of children's responses it was possible to infer the method of calculation the children employed, with the proportion of unclear responses ranging from .10–.19.

To consider why children's performance on direct relations problems was far below the ceiling level performance of the children observed in Study 3, the general performance of children in Study 3 is placed alongside the combined proportions of children from Study 4 who answered correctly or made an additive error (Figure 6.3). The reason for considering correct and additive responses together is that the additional requirement of interpreting a scalar relation can result from a conceptual misunderstanding that the relations are additive rather than multiplicative which would have led these children to the correct response for items in Study 3. Therefore by combining accurate performance with additive errors in Study 4 and comparing with the data from Study 3, the general level of discrepancy between the two studies can then indicate if interpreting the relation as additive is the main cause. Figure 6.3 shows the proportions of correct responses for children on direct extensive and intensive quantity problems from Study 3 and Study 4.

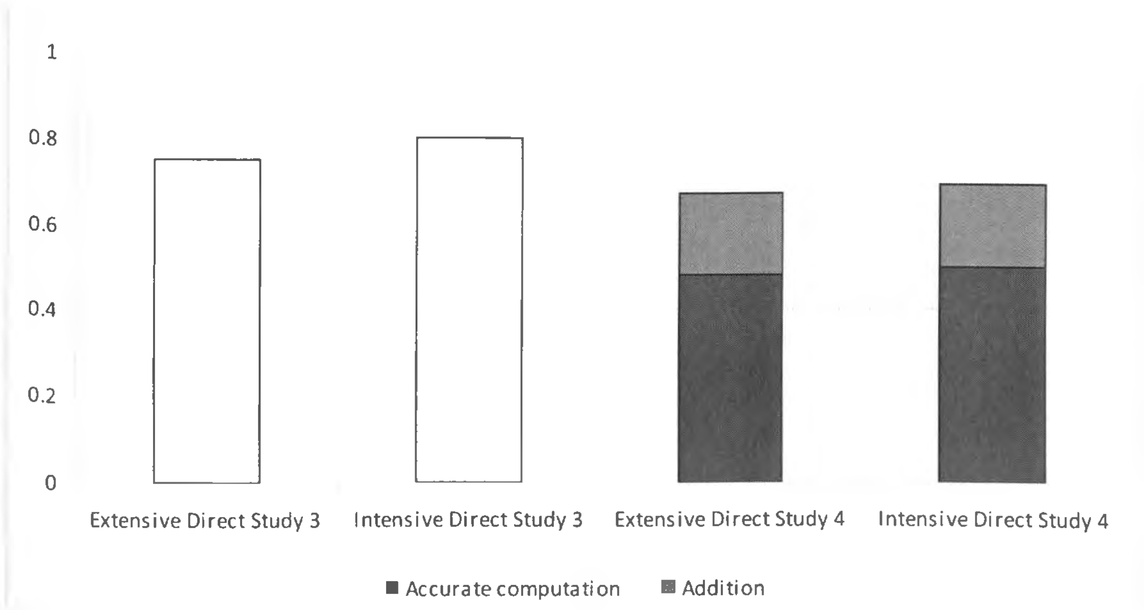


Figure 6.3 Proportion of reasoning success for children in studies 3 and 4 on direct relations problems

As can be seen in Figure 6.3, even when directionally accurate scores are combined, children's performance in Study 4 remains poorer than that of children in Study 3 on direct relations problems. These findings for direct relations items show that although asking children to perform a computation increased the difficulty of the problem, not all of this increased difficulty can be explained by the treatment of the scalar as an additive rather than multiplicative relation. Table 6.4 shows that children receiving computational problems produced another type of error not possible with the relational thinking problems of Study 3, which may explain the discrepancy in children's performance. For .17 of extensive and .14 of intensive quantity direct relations problems, children performed a multiplicative calculation with the wrong numbers. This suggests that for some children the additional requirement of computation leads children to lose track of the problem structure.

To explore why significant differences were observed between extensive and intensive quantity inverse relations problems during Study 3, but not in Study 4, the same type of analysis used on direct relations problems can be applied to inverse relations (see Figure 6.4).

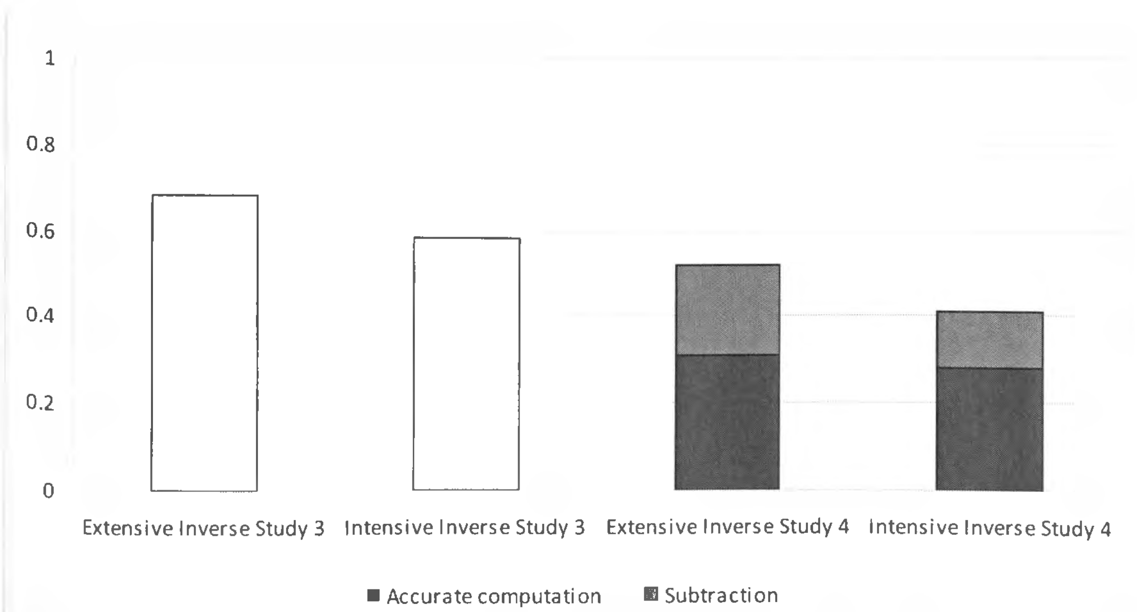


Figure 6.4 Proportion of reasoning success for children in studies 3 and 4 on inverse relations problems

In terms of computational accuracy, children performed at a similar level on extensive and intensive quantity items in Study 4, which was below that of the children from Study 3. If the subtraction strategy – which would have led to a correct reasoning pattern in Study 3, but not studying Study 4 – is added to the proportions of correct responses, then the pattern of results becomes interesting for two reasons. Firstly, even when children are given credit for correct directional reasoning, the children's performance in Study 4 still remains behind that of the children in Study 3. This suggests that the requirement of calculations generally had an adverse effect on the way children worked with inverse relations. Secondly, when accurate computations and subtraction errors are combined in Study 4, a striking similarity emerges in the pattern of results regarding the contrast between extensive and intensive quantity performance in the data from both studies. For children in Study 4, combining accurate computations with subtraction errors reveals that children's directional reasoning was more accurate in extensive quantity contexts. This suggests that although children have a better understanding of inverse relations in extensive quantity contexts, this may be mediated by the fact that children are more willing to interpret extensive quantity problems as additive relations than they are in intensive quantity settings.

6.4 Discussion

The requirement of children to perform calculations had a negative impact on children's general levels of success when compared to previous relational thinking studies conducted in the thesis. A significant effect of type of relation was observed, with direct relations items being solved more easily than inverse relations items. This led to the acceptance of the first experimental hypothesis which stated that direct relations items would be significantly easier to solve than inverse relations problems.

The additional requirement of having to perform calculations for the age groups tested led to comparable levels of performance for children on extensive and intensive quantity items. The failure to find a significant difference, and the relatively small effect size, means that it was not possible to accept the second experimental hypothesis which stated that extensive quantity problems would be significantly easier than intensive quantity items. For the age groups tested, presenting problems in extensive or intensive quantity settings contributed very little to children's computational accuracy.

The similarity between performances on extensive and intensive quantities problems was explored using an analysis of children's responses. As the children were taken from the same cohort as the children from Study 3, it was important to investigate why significant differences between extensive and intensive quantity problems on inverse relations did not occur in Study 4.

The exploratory analysis of children's responses showed that on inverse relations problems there were a higher proportion of children applying accurate reasoning strategies to inverse extensive quantity over inverse intensive quantity problems, though the children made the error of interpreting the relations as additive over multiplicative.

The point was made in the introduction to this study that much of the work conducted in the proportional reasoning literature is concerned with how children discover and then interpret scalar relations. By adapting Piaget et al's (1977) method it was possible to show

how an understanding of direct and inverse relations does not lead to the notion that they are multiplicatively related. This result is in line with Ferrandez-Reinisch's (1985) proposal that computational skill with inverse relations is preceded by a stage of awareness of inverse relations paired with confusion about how to compute the answer. Study 4 adds to this idea by showing that with extensive and intensive relational thinking problems, children who are aware of the direction of relations but unsure how to compute the answer tend to interpret the inverse relation as additive. It was also shown that the use of subtraction on inverse relations problems was more prominent in extensive quantity settings than intensive quantity settings.

In the literature on relational thinking, the argument is made that before solving inverse relations problems, children become aware of the importance of the second quantity but interpret the problem as direct (Squire and Bryant, 2003). Study 4 develops this argument by showing that when computations are required, a further step of interpreting the inverse relation as multiplicative appears. The additive interpretation of a given scalar extends the findings of Piaget et al (1977) beyond the isomorphism of measures structure to cover both the multiple proportion and product of measures structures. Study 4 also highlights that for some children the additional need to perform a computation in a direct relations problem causes them to lose track of the problem structure.

Although there were no systematic comparisons of extensive and intensive quantity computational problems in the literature, the point was made by Kaput and West (1994) that when proportional equivalence problems involve a third extensive quantity, children aged 11–14 years find computations easier. In Study 4 this was not observed for the direct and inverse relations problems as they were tested on a younger sample. This means that more one-to-one and perhaps longitudinal research would be required to understand more clearly how the use of extensive and intensive quantity problems interacts with children's computational success.

This study also highlights how, when looking at the beginnings of children's understanding of computations with multiple proportions relations, it is important to start with multiplicative relations where children are beginning to grasp the underlying logic. Strauss and Stavy's (1982) study of children's understanding of different relations important for the intensive quantity concept (replicated in Study 1), showed that children understood direct and inverse relations before sampling and proportional equivalence. The point was made in this chapter that children can solve direct and inverse relations problems without the need to understand the multiplicative relations between quantities. For both sampling and proportional equivalence problems, children need to be aware of the multiplicative relations. Study 4 provides a method of looking at younger children's awareness of the multiplicative relations between quantities within a logical framework with which they are familiar. Pursuing this method further could prove fruitful in understanding how children bridge the gap between a logical awareness of direct and inverse relations and a later understanding of proportional equivalence.

CHAPTER 7

DISCUSSION AND CONCLUSIONS

7.0 Overview

The aim of this thesis was to investigate how the type of quantity – extensive or intensive – affects children's developing understanding of multiplicative relations between quantities. A series of studies examined children's ability to solve direct and inverse relational thinking problems presented to them in extensive and intensive quantity settings.

The main focus of the thesis was an investigation of whether the distinction between extensive and intensive quantity relational thinking problems is conceptually important. There are two potentially significant features which distinguish intensive from extensive quantity relational thinking problems. Firstly, intensive quantity relational thinking problems always involve both direct and inverse relations, while extensive quantity relational thinking problems only involve inverse relations when problems involve the multiple proportions structure. Therefore it could be concluded that the distinction between direct and inverse relations explains much of the difficulty of intensive quantities. The second potentially significant feature which distinguishes intensive from extensive relational thinking problems is that with intensive quantity problems, the intensive quantity is expressed as a ratio between two independent quantities. This contrasts with the comparable extensive quantity multiple proportion problems which involve relations between three independent quantities. The question is whether this feature of intensive quantities contributes significantly to their difficulty.

The examination of conceptual difficulty with extensive and intensive quantity relational thinking problems was addressed by the experimental work of the thesis in three stages. The first stage (Chapter 3) considered the general intensive quantity concept and the connection between performance on direct, inverse, sampling, and proportional equivalence problems. There were three reasons why ideas important to the general intensive quantity concept were initially considered. The first was to verify the ages at which children solve inverse relations problems with intensive quantities. This was to select the most appropriate sample for comparisons between extensive and intensive quantity problems in the second and third stages of the experimental work. The second reason was to test whether the proposed methodology for comparing extensive and intensive quantities in later studies of the thesis – computer diagrams – impaired children's relational thinking about intensive quantities. As children experience many intensive quantities in their day-to-day lives, it was important to verify that children's thinking upon being shown computer diagrams was comparable to that of physical demonstrations which are more closely matched to what children may experience in an everyday encounter with an intensive quantity. The third reason was to examine the connection between the direct and inverse relational thinking problems studied in the thesis and other ideas important to the intensive quantity concept, namely operating on an intensive quantity (sampling) and understanding the relations between quantities as multiplicative (proportional equivalence).

The second stage of the experimental work focused on the distinction between extensive and intensive quantity non-computational comparison problems when direct and inverse reasoning was required (Chapters 4 and 5). Here the question was raised for the first time as to whether difficulties with relational thinking about intensive quantities are caused by the need to consider inverse relations, or whether understanding a quantity expressed as a ratio constitutes an added difficulty. This second stage of the experimental work also considered a second methodological question about whether the use of quantitative or relational descriptions of quantities on non-computational problems have an influence on children's thinking about relations between quantities.

The third stage of the experimental work explored further the comparison of extensive and intensive quantity problems involving direct and inverse relations. In this third stage the connection between children's knowledge of direct and inverse relations and an awareness of the multiplicative nature of these relations was considered by examining children's performance on extensive and intensive quantity computational problems.

The following discussion of the experimental work of the thesis is reported in three sections. The first section considers the contribution of the experimental work to the literature. The second section considers the educational implications of the thesis. The third section looks at the limitations of the thesis and suggests directions for future research.

7.1 Relational Thinking about Extensive and Intensive Quantities

The aim of the first section is to revisit the main conceptual themes set out in the introduction and literature review and to place the results of the current thesis within the context of the established literature. There are four themes considered in this section; (1) direct and inverse relations and the development of children's understanding of the wider intensive quantity concept; (2) the impact of different forms of problem presentation on relational thinking success; (3) the classification of the intensive quantity structure within Vergnaud's theory of multiplicative structures; and (4) the reasons why relational thinking about quantities is difficult.

7.1.1 Direct and Inverse Relations and the Development of the Intensive Quantity Concept

The issue of how children develop an understanding of the intensive quantity concept had been approached most systematically in the literature by Strauss and Stavy (1982). They considered different types of relations between quantities problems needed to build the intensive quantity concept. Using sweetness Strauss and Stavy tested three 'relations between quantities' problems (direct, inverse and proportional equivalence) and a second type of problem (sampling) in which children had to think about the result of an operation on an intensive quantity. In Strauss and Stavy's original study on sweetness, the direct relation was solved more easily than the inverse relation and the inverse relation was acquired earlier than either sampling or proportional equivalence.

Strauss and Stavy's conclusions on whether the precise relational thinking needed for proportional equivalence problems was more or less difficult for children than thinking about the result of an operation on an intensive quantity with sampling, were less clear. It was reported that before the age of 8 years, sampling problems were easier to solve than proportional equivalence problems. At around 8 years, as sampling problems were solved less successfully by children, both sampling and proportional equivalence problems were of equal difficulty. By 9 years, children began to be more successful with proportional equivalence problems, while making less progress with sampling.

Much of the difficulty in trying to relate Strauss and Stavy's (1982) findings with sweetness to the more general debate on the development of an understanding of intensive quantities stems from the fact that sweetness only represents one part of the taste concept. Strauss and Stavy's study did not adequately control for the language used to represent taste and this limits the scope of their research. An additional problem with Strauss and Stavy's study was that they only presented children with a single problem on each of the four problem types. The consequence of adopting this research design is that for sampling and proportional equivalence problems in particular, it is difficult to establish the extent to which the success of younger children was due to chance. These sampling and proportional equivalence problems were particularly susceptible to chance responding because the question always posed to children was 'will these drinks taste the same?'. For sampling and proportional equivalence problems this means that children who were guessing had a 50% chance of passing the item.

It was the specific aim of the second experiment reported in Chapter 3 to extend Strauss and Stavy's (1982) initial inquiries on the development of the sweetness concept to address the more general question of intensive quantity development. This aim was addressed by working with taste rather than sweetness as the core concept. By looking at taste it was possible to control for the type of language used, producing a design focused on the intensive quantity concept. To reduce the impact of chance responding adding noise to the data on sampling and proportional equivalence problems, Strauss and Stavy's research strategy was modified in two ways. The first was to increase the overall number of problems children were asked to solve. The second modification was to analyse the data using a partial credit Rasch model. The use of the partial credit Rasch model allowed children's relative success to be judged on the basis of combined judgement/justification scores, for which children were given more credit for correct justifications than for correct judgements.

With this revised research strategy, the order of problem difficulties reported by Strauss and Stavy (1982) was shown to hold up well with the same order of relative difficulty produced for direct and inverse relations, both of which were easier than sampling or proportional equivalence problems. From these results it was possible to verify the ages at which children solve direct and inverse relations problems with intensive quantities. This helped to decide the target ages of the extensive and intensive quantity comparison studies later in the thesis. For the issue of whether children can solve sampling problems earlier than proportional equivalence problems, it was shown that once the possibility of chance responding is reduced, and children's justifications are taken into consideration, sampling problems are conceptually easier than proportional equivalence problems. Sampling problems were shown to be just at the upper reaches of the ability levels of the most able group tested in Chapter 3 (9 year olds). Proportional equivalence problems were shown to be beyond the ability of the children tested in Chapter 3.

7.1.1.1 The Difficulty of Proportional Equivalence Problems

In order to explain why proportional equivalence problems were the most difficult of the logical relations, it is suggested that proportional equivalence is the only problem which requires an explicit understanding that the relations between quantities with intensive quantities are multiplicative. Direct and inverse relations problems can be solved through reasoning either 'more is more' (direct) or 'more is less' (inverse), regardless of the differences in magnitudes between the quantities involved as the presence of a difference is the important factor. For sampling problems, the challenge is to understand that the operation of sampling an intensive quantity does not change the taste, which is quite different to the computation of equivalences. Even though the proportional equivalence problems in the current thesis only required consideration of simple proportional increases (half vs. full cups), these problems remained difficult for all the age groups tested. Therefore, the problem for children was not the multiplication procedure, as only a simple awareness of doubling was needed; instead the problem was understanding that the quantities were related multiplicatively. The plausibility of this explanation received support from Study 4 (reported in Chapter 6). Here it was shown that when children needed to be aware of the multiplicative nature of the relations between quantities to solve direct and inverse relations problems by performing simple calculations, the relative difficulty of even the most basic direct relations problems increased.

7.1.2 The Impact of Different Forms of Problem Presentation on Relational Thinking Success

7.1.2.1 How Variations in Problem Presentation Contribute to Relational Thinking Success with Non-Computational Problems

The proportional reasoning literature provides many examples of how the form of problem presentation can have an impact on children's success. For the current thesis, a collection of problem presentation methods were employed in order to assess their impact on children's ability to solve relations between quantities problems. The collections of different problem presentations were chosen primarily from the literature on sampling and joining intensive quantities. The aim was to explore whether variations in problem presentation would produce significant differences on direct and inverse relations between quantities problems. Five variations of problem presentation were investigated. The first and second forms of presentation were taken from Schwartz and Moore's (1998) study, from the sampling of intensive quantities literature, which varied the presentation of sampling problems using realistic or diagrammatical displays of problems. The third and fourth forms of presentation were taken from the often reported findings in the joining and sampling literature on the influence of quantitative and relational information on children's thinking (Cowan and Suttcliffe, 1991; Harel, Behr, Lesh and Post, 1994; Desli, 1999; Jäger and Wilkening, 2001). The fifth form of presentation examined in the thesis was taken from Tourniaire and Pulos' (1985) study from the wider proportional reasoning literature, which found that manipulatives increased children's proportional reasoning success. The use of manipulatives was included because Tourniaire and Pulos reported that manipulatives increased the success rates of children with low proportional reasoning ability.

The impact of varying problem presentations on children's success with relations between quantities problems in the current thesis proved limited. When comparing children's performance on taste problems presented either by physical demonstrations or by computer diagrams (Study 1), the lack of significant difference in performance levels provided a valuable methodological clarification that presenting intensive quantity problems in the remaining studies of the thesis using computer diagrams would not alter children's thinking. This means it was possible to compare children's performance across both extensive and intensive quantity settings.

The most notable difference in performance arising from the various forms of presentation tested appeared between problems presented with quantitative information and problems presented with relational information (Study 3). In both extensive and intensive quantity settings, the children who received problems with quantitative descriptions of the quantities found inverse relations problems more difficult than the children who received problems with relational terms only. The poorer performance on inverse relations problems of children receiving quantitative descriptions coincided with an increased likelihood of these children offering 'the more, the more' explanations.

7.1.2.2 The Impact of Quantitative Descriptions on Relational Thinking

The point was made in the review of the literature that in the attribution theory literature (Kun, 1977; Suber, 1980), children were more reluctant to infer the inverse relation 'more effort means less ability', than 'more ability requires less effort' due to the cultural beliefs of the children tested. One possibility for the increased frequency of direct relations errors when inverse relations problems are presented with quantitative descriptions is that it may reflect an educational bias towards treating numerical relations as directly related. It is possible that in the child's experience of working with numerical information, direct relations are more prevalent than inverse relations. Indeed this is what was found in the review of the literature when looking at the National Numeracy Strategy (DfEE, 1999, 2003), which the children participating in the current thesis had followed and which consisted mainly of directly related isomorphism of measures problems. Nesher (1988) has also shown that educational experience dictated the type of problem spontaneously produced by children. So it is quite possible that children's educational experiences, which favour direct numerical relations, influence their thinking when receiving quantitative descriptions.

The increase in direct relations solutions given when relational thinking problems were presented with quantitative descriptions also supports Resnick and Singer's (1993) proposal that relational knowledge and knowledge of numbers initially develop independently. Resnick and Singer argue that when children think about numerical relations they often resort to direct relations interpretations, as children have more experience with additive problems. This would explain why the children who received the problems with relational descriptions in Study 3 made fewer direct relations errors.

One difficulty with accepting this interpretation of the results is that in Study 3 the children's performance on direct relations problems was at ceiling level, meaning that only performance on inverse relations could be interpreted. To pursue Resnick and Singer's hypothesis further and test whether the use of quantitative descriptions of quantities leads to a general increase in direct reasoning would require working with a wider age range than was used in Study 3.

7.1.2.3 The Effect of Computational Problems on Relational Thinking Success

The next question raised in the thesis was whether children's success with relational thinking problems was affected by the additional requirement of performing an operation. There are reports in the intensive quantity literature on relational thinking on how varying the numbers in a problem has a limited impact on children's success (Piaget and Inhelder, 1974; Squire and Bryant, 2003). An examination of children's ability to solve calculation problems about direct and inverse relations in extensive and intensive quantity settings had not previously been approached systematically in the literature.

In the wider proportional reasoning literature the question of computational problems had focused on whether making scalar or functional relations easier to work with influenced children's computational success (Vergnaud, Rocchier, Ricco, Marthe, Metregiste and Giacubbe, 1979; Noelting, 1980; Karplus, Pulos, and Stage, 1983; Christou and Philippou, 2002). As the thesis looked specifically at children's ability with direct and inverse relations rather than computing proportionally equivalent relations, the question of computations is different from the traditional missing value problem. In the current thesis, the impact of computational problems on relational thinking was explored by formulating problems in which the scalar relation was given. The children had to interpret this relation to find the value of a missing quantity. Although this type of problem has not received specific attention in the intensive quantity literature, the method employed in Study 4 extended the method originally reported in Piaget, Grize, Szeminska, and Bang (1977) when working with direct relations between two extensive quantities (eels and food). In Piaget et al's study, children were shown to progress from interpreting the scalar as an additive relation to a multiplicative relation systematically by around 10 years. Piaget et al.'s (1977) method was extended by creating problems in which children had to compute both direct and inverse relations. As the computations that the children were required to make were designed to be easy, it was expected that the overall performance would be lower than that of the children who received relational thinking only problems (studies 2 and 3). The reason for this being that computational problems require an increased the level of accuracy required over comparison problems for which only three choices are available. As the problems were designed with the need for only simple computations the children were still expected to perform well.

The reports in the literature suggested that children's responses could fall into any of three categories. Firstly, correct answers, this is based on the argument made by Vergnaud (1982) about relational calculus with additive structures. He stated that once children understand the relation between quantities, the arithmetic is simple. If Vergnaud's suggestions with additive structures are also true for multiplicative structures, then the additional requirement of a simple computation should make little difference to the results. For the computational problems tested in Study 4 the proportion of correct responses was approximately 50% for direct and 30% for inverse relations problems. This means that these children understood the relations and also found the computations simple, thus following Vergnaud's predictions. It must be noted though these levels of success were lower than those of children taken from the same cohort who had solved non-computational relational thinking problems in Study 3. Even on direct relations problems, the additional requirement of even simple computations made the problems more difficult. This means that Vergnaud's ideas with additive structures did not transfer easily to the multiplicative structures tested in the thesis. For some children, understanding the general direction of the relation between quantities and arithmetic success are separate challenges with multiplicative structures.

This was highlighted further by the second category of expected response, following Piaget et al. (1977). In this category were the responses of children who knew the direction of the relation but applied additive rather than multiplicative solutions. The analysis of children's responses showed that 20% of responses on direct relations problems and 13% on inverse relations problems could be explained in this way. The interesting point here is that even though these children had a good notion of the relations between the quantities in the problem and they had been taught the necessary operations needed to make accurate computations, they still made conceptual mistakes regarding the operation needed. This shows that when children are beginning to think about relations between quantities there is a difficulty accommodating these ideas within an emerging concept of multiplication. Additive errors further highlight why understanding general relations between quantities does not necessarily lead to arithmetical success. The children who used additive strategies may have understood the relations but were still unclear about the boundaries of additive and multiplicative structures, which led to additive computational errors.

The third category of expected response based on the established literature contains the responses of children who assumed the wrong direction of the relation between quantities (Stavy and Tirosh, 2000). As with the non-computational studies (2 and 3), this was not observed in children's responses to direct relations problems, though for around 30% of responses to inverse relations problems, children computed using multiplication. These three categories of responses cover what may be expected, following a review of the relevant literature. For children's computations on inverse relations problems the three categories covered children's responses well. With computations on direct relations problems children exhibited a fourth and unexpected category of responses.

For around 15% of the children solving direct relations problems, the correct operation (multiplication) was performed but the children's computations involved the wrong numbers. This group is interesting as the multiplication of the wrong numbers in the problem indicates that the additional computation demands of the problem led these children to lose track of the structure of the problem. The interference of a mathematics test setting on children's reasoning has been shown before in Harel, Behr, Lesh and Post's (1994) sampling study. Here children did poorly on sampling items, often attempting computations, as the sampling items were part of a more general mathematics test. It could well be the case that some of the children solving the computational problems in Study 4 reacted to the numbers purely as a calculation exercise and therefore paid little attention to the conceptual structure of the problem. It is not possible to draw a firm conclusion about why this would lead some children to perform computations detached from the problem structure as the study was not designed to test this question.

If children were reacting to the problems as a computation exercise then it would be interesting to see what determined their choice of calculations, as performing multiplication with any of the numbers would also lead to correct responses on occasions. One possibility is that these children did not really understand the relations between quantities and in the face of instructions to compute, performed the multiplication they found the easiest or recalled a known number fact. Understanding the performance of these children better would require designing a study in which relational thinking skill was assessed independently from computational success. This would provide a good baseline measure of the children's relational thinking ability. The children could then be interviewed while solving computational problems to understand the reasons behind their computational choices. Pursuing this question would help to clarify how children's relational thinking success and computational choices interact.

7.1.3 The Classification of Intensive Quantity Problems within Vergnaud's theory of Multiplicative Structures

This question of whether intensive quantities represent a unique problem structure follows a debate played out in psychology of mathematics education literature, where alternative ways of classifying multiplicative problems have been presented in the theories of Vergnaud (1983) and Schwartz (1988). Although the more detailed of the two, the classification theory provided by Vergnaud was shown to have two potentially significant issues unresolved. The first was that it did not discuss the issue that, within the multiple proportions structure, quantities can be inversely related. The second unresolved issue is whether Vergnaud's multiple proportions structure is sufficient to include both intensive and extensive quantity problems.

7.1.3.1 Is the Distinction Between Direct and Inverse Relations Important?

The first issue of whether the distinction between direct and inverse relations represents a significant conceptual challenge for children was raised in the literature review. The literature shows that with culturally specific dimensions such as ability, effort and outcome, direct relations are easier to think about than inverse relations (Karabenick and Heller, 1976; Kun, 1977; Surber, 1980). With relations between logico-mathematical concepts, such as the extensive and intensive quantities under investigation in this thesis, the picture is less clear. Although there are reports in the literature which indicate that the distinction between direct and inverse relations is significant (Chapman, 1975; Wilkening, 1981; Strauss and Stavy, 1982; Acredolo, Adams and Schmid, 1984), in most cases these studies contain methodological problems making it difficult to draw firm conclusions from these results.

Therefore, examples from Piaget's work were used in the literature review to establish a general reference to the distinction between direct and inverse relations, as Piaget's experiments were designed to look specifically at the development of relations between quantities, independently of perception. Piaget's experiments produced inconsistent results, as Piaget's stance on the importance of distinguishing between direct and inverse relations changed during the course of his career. Initially, Piaget viewed the distinction between direct and inverse relations as important (Piaget and Inhelder, 1958), though he later placed the acquisition of direct and inverse relations at roughly the same age (Piaget, 1970; Piaget and Inhelder, 1974).

However, Desli (1999) looked specifically at children's performance on direct and inverse relations problems with intensive quantities and showed that direct relations were easier than inverse relations. The experiments in the current thesis were designed in part to strengthen Desli's argument about the important distinction between direct and inverse relations problems.

The first experiment in Study 1 showed that both direct and inverse relations with intensive quantities are difficult for children under the age of 6 to solve systematically, which is in line with what is known about extensive quantities (Correa, Nunes and Bryant, 1998; Sophian, Garyantes and Chang, 1997) while by 9 years children are able to handle both direct and inverse relations quite well. Study 2 looked at the distinction between direct and inverse relations across a wider range of intensive quantity contexts than those previously used by Desli (1999). The type of problem structure under consideration was also extended to include extensive quantity multiple proportions problems. In both intensive and extensive quantity contexts, significant differences in children's performance on direct and inverse relations problems were observed. Study 3 replicated these results with older children.

As the first three studies had focused on the distinction between direct and inverse relations with non-computational comparison problems, Study 4 looked at this distinction with computational problems. The general performance of the children was lower than that observed on non-computational problems, though the general pattern of results remained, with greater overall success reported on direct than on inverse relations problems. The results reported in the thesis consistently showed that for children specifically between the ages of 7 and 9 years, the distinction between direct and inverse relations is important and produces meaningful variations in children's performance.

On the question of direct and inverse relations, the general conclusion that can be reached supports earlier work by Desli (1999), which stated that in the development of children's understanding of relational thinking about quantities, there is an important distinction between direct and inverse relations. The findings from the sharing literature (Correa, Nunes and Bryant, 1998; Sophian, Garyantes and Chang, 1997) were also extended to include a wider range of extensive quantity settings.

7.1.3.2 The Distinction Between Extensive and Intensive Quantity Relational Thinking Problems

The second potentially significant unresolved issue from Vergnaud's (1983) work on multiplicative structures is whether intensive quantities have a role in his theory. To test the possibility that intensive quantities present a unique challenge, children's performance on intensive quantity problems was compared to that of comparable extensive quantity multiple proportion problems. It follows that when comparing extensive and intensive quantity problems, if the only difficulty in understanding intensive quantity relational thinking problems is an awareness of the inverse relation, then no significant differences should exist between extensive and intensive quantity problems. If the dependence of quantities in the intensive quantity structure represents an added difficulty, then significant differences between extensive and intensive quantity problems should appear.

Even though extensive and intensive quantity problems had not been compared systematically in the literature, Kaput and West (1994) suggested that computational problems are easier for children when three extensive quantities are involved. Kaput and West's prediction was made on the basis of the post-hoc analysis of children's success on a problem relating quantities of china, silverware, and placemats. Kaput and West argued that because children could use the quantity 'placemats' to hold the quantities of china and silverware together while each quantity retained a separate identity, the placemats problem was easier than the dressing problem they posed, in which the ingredients used to produce the taste lost their separate identities in the process.

For the relational thinking studies (2 and 3), which provide the first systematic tests of this distinction, significant main effects of quantity were observed. In both studies extensive quantity (multiple proportions) problems were significantly easier for children than intensive quantity problems. Although significant differences were observed between extensive and intensive quantities on non-computational problems, no significant differences were found with the comparison of computational problems in Study 4. One possibility for a lack of significant difference on computational problems is that the children (aged 7–9 years) were chosen to be on the boundaries of solving non-computational problems. The additional requirement of computations made all the problems too difficult for these children regardless of whether they involved extensive or intensive quantities. The analysis of children's responses to the computational problems confirmed this. It was shown that although computational success was comparable on extensive and intensive quantity problems children's general awareness of directional reasoning was better in extensive than intensive quantity settings.

On the basis of the results reported in studies 2 and 3, it is argued that when children begin to understand inverse relations, the intensive quantity structure represents a unique conceptual challenge for children. This is due to the importance of conceiving of a quantity which is expressed as a ratio of previously unrelated measures. The inclusion of an intensive quantity in the problem was shown to constitute an added difficulty for children beyond the challenge of the traditional multiple proportions structure.

It could be that the extensive quantity multiple proportions problems tested in the current thesis were easier than intensive quantity problems because the extensive quantity problems required children to think about the result of an operation. For example, when a number of pigs eat a quantity of food over a number of days, the operation 'feeding' may help children to organise their thinking. The result of an operation as a factor increasing children's ability to work with inverse relations would be consistent with the sharing literature where young children were successful in working with inverse relations. More work is needed though to develop this hypothesis further.

7.1.3.3 Developing Vergnaud's Theory of Multiplicative Structures

The main contribution to the classification of multiplicative structures literature from the work carried out in the current thesis is that the omission of inverse relations and the intensive quantity comparison structure from Vergnaud's theory of classification leaves the theory incomplete. The recommendation of the thesis is not that Vergnaud's theory should be rejected but that it should be developed to accommodate the current findings. This is because despite it's shortcomings, the structures originally developed by Vergnaud have greater explanatory power than those proposed by Schwartz (1988). Schwartz's theory for example does not capture the important distinction between product of measure and multiple proportion structures for which empirical evidence exists (Vergnaud, Ricco, Rouchier, Marthe, and Metregiste, 1978). There is also no room in Schwartz's theory for the distinction between isomorphism of measures problems and the intensive quantity comparison problems used in the current thesis as Schwartz's (1988) definitions of intensive quantity is too general. As making Vergnaud's theory more complete requires the expansion of his current structures rather than a major restructuring, as would be the case with Schwartz's theory, Vergnaud's theory is therefore preferable. The two changes recommended by the thesis to Vergnaud's theory are firstly the inclusion of inverse relations to the multiple proportions structure. It was shown consistently throughout the studies that in extensive quantity multiple proportions settings the distinction between direct and inverse relations was significant.

The second change required to increase the descriptive power of Vergnaud's multiplicative structures is to find a suitable place for intensive quantity problems. As intensive quantity problems differ from extensive quantity problems in terms of difficulty, one possibility would be to include intensive quantity problems in the product of measures structure. The difficulty with this is Vergnaud's product of measures category was formulated to be for quantities which arise as a result of multiplication of quantities (e.g. number of shirts × number of shorts = number of possible outfits), whereas intensive quantities are formed as the quotient of two quantities. The key similarity between Vergnaud's product of measures structure and intensive quantities is that the result of the product calculation and the intensive quantity are new quantities dependent on two separate quantities. This contrasts with multiple proportions problems where each quantity remains independent.

The solution may be to rethink Vergnaud's structures slightly to develop a structure which can accommodate both product of measures and intensive quantity problems. To achieve this it may be wise to return to the term 'multiplication of relations' used by Piaget (1946). For Piaget, the term multiplication of relations referred to multiplicative reasoning (which could result from a multiplication or division operation), which has a broader reach than Vergnaud's product of measures structure, which is dependent on the operation of multiplication. If the term multiplication of relations is adopted, then it is possible to place product of measures and intensive quantity problems into the higher level classification of multiplication of dependent relations problems. This would capture the important similarity that in both product and intensive quantity problems one of the quantities is dependent on the others. It would also be possible to use the term multiplication of independent relations to refer to Vergnaud's multiple proportions structure in which several independent quantities are related multiplicatively.

More work is needed to develop this argument, specifically on the distinction between extensive multiple proportion and intensive quantity problems requiring computations, which were too difficult for the children tested in the thesis. There is sufficient evidence though from the comparison of children's performance on non-computational relational thinking problems to suggest that this question is worth pursuing.

7.1.4 Why is Relational Thinking Difficult?

The final question raised in the literature review is; why is relational thinking about quantities difficult for children? The literature has provided a variety of explanations. The most relevant to the current thesis are represented by the work of Stavy and Tirosh (2000), Noelting (1980), and Correa, Nunes and Bryant (1998).

Stavy and Tirosh (2000) presented the general argument that children's performance on many scientific and mathematics tasks is governed by the use of two intuitive rules: 'the more A, the more B' and 'the same A, the same B'. In the context of the current thesis, Stavy and Tirosh's intuitive rules theory would predict that children should apply an intuitive rule as a blanket response to relational thinking problems they find difficult. Children using intuitive rules should therefore solve direct relations problems ('the more A, the more B') more easily than inverse relations problems for which no intuitive rule exists. In order to state that Stavy and Tirosh's theory explains why relational thinking problems are difficult, it follows that direct relations should always be easier than inverse relations. It is important to note that with Stavy and Tirosh's theory the use of intuitive rules rather than what different rule use might indicate is of greater importance.

Noelting's (1980) work on proportional reasoning with taste problems provided a different perspective, as Noelting was more concerned with the development of proportional reasoning competence. On the question of solving direct and inverse relations problems Noelting suggested that children would initially reason about the directly related quantity ignoring the inversely related quantity. This is similar to the early reports of Piaget (1946) on the conservation of continuous quantities, where children would pay attention to one dimension only. Noelting reports that when the child realises the importance of the second quantity the child is able to solve inverse relations problems. For Noelting's ideas to explain why relational thinking is difficult, it would be expected that children would perform well on directly related problems but fail inverse items, by focusing on the wrong quantity and giving 'the same A, the same B' responses.

More recent work by Correa, Nunes and Bryant (1998), later replicated by Squire and Bryant (2003), has suggested the transition from solving direct to inverse relations is not as straightforward as reported in Noelting (1980). Working with sharing and division, Correa, Nunes and Bryant (1998) showed that younger children tended to make inconsistent errors on inverse relations problems. It was also shown that older children made the systematic error of applying direct relations to inverse relations problems as a result of trying to incorporate both quantities into the solution. The prediction that can be made from Correa, Nunes and Bryant's (1998) work is that in trying to coordinate two variables simultaneously children will make direction of relation errors.

The results reported in Study 2 produced a pattern of responses for which intensive quantities direct relations problems were not always easier than inverse relations. Although in general children scored higher on direct vs. inverse relations, speed and cost problems showed the opposite result. This pattern of results was also replicated in Study 3. This means that for the development of intensive quantity relational thinking the intuitive rules theory proposed by Stavy and Tirosh (2000) did not predict or explain children's performance.

The analysis of children's responses to relational thinking problems in Study 2 was undertaken to look at whether children displayed a systematic error pattern which would indicate a specific conceptual challenge when working with relational thinking problems. The main question was whether the reason behind children's errors on intensive quantity relational thinking problems was due to a difficulty in thinking about the direction of the relation, as had been observed by Correa, Nunes and Bryant (1998) in sharing problems, or whether children would reason on the basis of one quantity only as suggested by Noelting (1980). The results reported in Study 2 showed that although within particular intensive quantity settings children produced systematic error patterns, the type of error varied between problems. For example when children solved intensive quantity problems using terms such as 'sweetness', 'more slowly', 'more/less expensive', 'more crowded', children tended to reason about one quantity only. When asked using terms such as 'value', 'quicker', and 'sourness', direct relations errors were more prevalent. The follow up study reported in Study 3 showed that with the older sample tested, children's errors on inverse relations problems were more likely to be caused by the over-application of the direct relations (p.161). This appearance of direct relations errors on inverse relations problems was not predicted from Noelting's descriptions.

The results of the thesis lead to the conclusion that children find relational thinking about quantities difficult for two reasons. First of all, children must understand that both quantities are important (as first shown by Piaget (1946) in his conservation of continuous quantities studies). Secondly, children must understand the direction of the relation between the second quantity and the outcome, this supports Correa, Nunes and Bryant (1998) and Squire and Bryant's (2003) reports on sharing problems.

The notion however that children work with direct relations before inverse relations was shown to be an oversimplification of children's thinking. There remain questions as to why in some cases the inverse form of a problem was found easier than the direct form.

7.1.4.1 When do Children Find Inverse Relations Easier Than Direct Relations?

When the intensive quantity problem involved speed or when cost problems were presented with terms such as 'better value', children were shown to score higher on the inverse form of the problem. Although evidence of younger children using inverse relations successfully is not new (Correa, Nunes and Bryant, 1998; Sophian, Garyantes and Chang, 1997), what is unique about these findings with speed (studies 2 and 3) and 'better value' (Study 2) is that children were more successful in solving the inverse than the direct form of the problem.

One possibility for this pattern of results is that there are examples of intensive quantities terms, like 'quickly' and 'slowly' which hold a very specific meaning in the child's culture, meaning that children only think about one possibility. For example children may draw on their vast perceptual knowledge of speed, whether it is from racing against friends in the playground, sports days or watching motor racing on TV. In these contexts children need only to think about speed in relation to the finishing time, as the control of distance is a given convention of the race setting. For 'better value', the same thing could be happening with children more accustomed to value as an absolute (how much money will it cost) rather than relative problem of spending the same amount of money but getting more for it. For example the common place supermarket promotion of 'buy one, get one free' suggests that the cost of a product has not changed, but that the supermarket is giving a reward for buying a particular product. So although the actual price paid for the product is the same, the relative cost is lower. In short, speed and cost may be genuine examples of where children's informal cultural knowledge favours the inverse relation.

A second possibility is that these examples with speed and cost demonstrate that the distinction between direct and inverse relations is an artefact of a wider issue that, for children thinking about relations between quantities, is all about variable salience. For example, if children are asked about sweetness, then they will think about the amount of sugar and not the amount of water. Likewise, when children are asked about wateriness of taste, children would only pay attention to the amount of water. In most cases this form of variable salience would lead children to the correct answer on direct relations problems but not on inverse relations problem. If time and money are the salient variables of speed and cost respectively, then it could lead to a greater awareness on inverse relations problems.

For this second interpretation to be more plausible, and if relational thinking success is determined by variable salience, then it follows that children should reason on the basis of the salient variable in both direct and inverse problem settings. Using the example of sweetness again, children only thinking about sugar would correctly respond that the more sugar means more sweet taste on the direct relations problem. On the inverse relations problem sugar as the salient variable would lead children to reason that the same sugar would mean the same level of sweetness.

The results obtained across the thesis provide more support for the first interpretation than for the second: that in isolated cases cultural conventions lead children to particular answers. The reason why a wider difficulty of variable salience was not supported by the results in the current thesis is found in the general patterns of children's errors. The most common error type on inverse relations problems, reported across the relational thinking studies (2 and 3), was an overuse of direct relations. For example, with the sweetness problem in Study 3, children were more likely to produce the 'more water means a more sweet taste' error on the inverse sweetness problem. On problems involving speed and value, where salience can explain children's performance, the question of why the second quantity does not figure in children's thinking is outstanding.

There are two reasons why cultural conventions can lead children to focus on one variable when solving relational thinking problems, which are captured by Bereiter and Scardamolia's (1982) distinction between conceptual and functional limitations in children's thinking. Conceptual limitations refer to the idea that the child will fail to consider a variable in their solutions as they do not know that the quantity is significant. A functional limitation would occur when the child is aware of the second quantity but does not use it in problem solving. The important difference is that with a functional limitation, the child knows the quantity is significant but is unable to do anything with the variable, so it plays no functional part in the child's problem representation. Indeed discussion of children's strategies for dealing with functional limitations of inverse relations problems was developed by Squire and Bryant (2003) when they talk about children's use of direct relations to solve inverse relations problems. This use of direct relational thinking on inverse relations problems was also shown in the current thesis on problems presented in chapters 4 and 5. The question of whether use of 'the same A, the same B' strategies represents an even earlier type of functional limitation, or whether it is a conceptual limitation was beyond the scope of the thesis.

7.2 Educational Implications

To work with proportional relations children must understand that relations between quantity problems are determined by more than one quantity. Before children can think about equivalent proportions they must develop an awareness of direct and inverse relational logic. This logical necessity identified by Inhelder and Piaget (1958) was supported by the first experiment of the thesis, Study 1, which extended Strauss and Stavy's (1982) original work to include more systematic language controls. It was also shown, during chapters 4–6 (studies 2–4), that children's acquisition of relational logic precedes children's procedural success. The educational message from this collection of findings is clear; introducing children to proportional relations between quantities can highlight the important conceptual distinctions that exist between addition and multiplication. For example, the inverse relations that occur with relations between quantities problems reflect how multiplicative relations work differently to additive relations. As Schwartz (1988) reports, the multiplication of quantities does not always lead to a bigger quantity (e.g. more time, less speed). Although Vergnaud (1983) highlighted the procedural similarities that exist between multiplication and addition through the scalar solution to isomorphism of measures problems, relations between quantities problems should be used to create important logical distinctions. For this reason the teaching of relations between quantities should begin by extending children's relational awareness rather than focusing on procedural accuracy. It is unfortunate that the current version of the numeracy strategy in the UK does not include any specific guidance on the teaching of the distinction between direct and inverse relations between quantities. As was shown in the review of the literature, relations between quantities situations, such as cost, do appear in the numeracy strategy but only in what would be regarded as an isomorphism of measures context (to use Vergnaud's classifications). For example, the inclusion of a cost problem is done more to present multiple isomorphism of measures contexts to keep children interested than in order to develop children's conceptual understanding of the quantities related to cost. Although the isomorphism of measures structure represents the simplest multiplicative structure, it must also be remembered that the isomorphism of measures structure only works in terms of direct relations. The omission of the explicit teaching of inverse relations between quantities from the English curriculum as it exists in other education systems, e.g. China, Taiwan and Japan (Lo, Cai and Wantanabe, 2001), and more specifically, intensive quantities from the primary curriculum, is to miss the opportunity to promote in children both the logical distinction between multiplication and addition, and the important structural similarities that exist between many concepts that they will rely on in later science learning and in their day-to-day lives.

Although the current thesis suggests that children should be introduced to relations between quantities problems, the best way in which to achieve this is not entirely clear. Intervention work on whether it is possible to teach children about inverse relations in one intensive quantity context in the hope that they will transfer it to other was conducted by Stavy (1981). In Stavy's experiment, 40 intervention children (7–10 years) received two hours of explicit teaching about inverse relations about salt water across two science lessons. Even after two hours of explicit teaching, half of the intervention group showed no improvement (even on the target question) between pre- and post-test while only 9 of the children who showed improvement were able to transfer their understanding of the inverse relation across the three contexts used at post-test. These results were quite disappointing considering the transfer problems (rate of cooling, taste of chocolate bread) were composed to be visually similar to the target salt/water problems.

The results of the thesis suggest two alternative approaches which may prove more successful, though future research is needed to test which method is more effective. The first way to introduce children to the idea of relations between quantities is through the multiple proportions structure. Although the intensive quantity concept covers many of the quantities children meet in their day-to-day lives, the empirical evidence presented in chapters 4 and 5 (studies 2 and 3) shows that children understand relations between three independent quantities sooner than they do with intensive quantities, when a quantity is expressed as a ratio. Whether teaching children about the multiple proportions structure, which requires thinking about the result of an operation, can help children organise their thinking about relations between quantities. Further investigation into whether this can then be used as a way to promote relational thinking about intensive quantities would be warranted.

A second way to introduce children to relations between quantities problems would be to teach intensive quantities as a collection of related concepts. This is different to intervention studies with intensive quantities in the literature which tend to focus on a particular intensive quantity (e.g. Thompson, 1994; Fujimura, 2001). Teaching intensive quantities as a set of related concepts would provide children with the opportunity to explore the ways in which known situations with different problem settings actually behave in very similar ways (structurally). This approach could help children develop a more rounded idea of relational thinking. Intensive quantities may be preferable to introduce relations between quantities problems, as it was shown in chapters 4 and 5 (studies 2 and 3) that children developed an understanding of the inverse relations related to speed earlier than the direct relation, while for other intensive quantities (e.g. taste, density) the direct relation was easier. By teaching children about the intensive quantity concept it could be possible to explain the idea of inverse relations in taste by using speed as an example or vice versa. Intervention work of this type, with the goal of developing many aspects of the intensive quantity concept in children aged 9–11 years was carried out by Howe, Nunes and Bryant (2004). Children received interventions across four twenty-five minute lessons, with the first lesson designed to promote children's understanding of inverse relations, across intensive quantity contexts. Post-test results showed that children made significant improvement on their solutions and explanations to intensive quantity problems. Although this project was carried out with children older than those participating in the current thesis, and was carried out with more ambitious goals than simply to teach children the inverse relation, their results suggest that teaching the intensive quantity concept is a promising avenue to pursue in the promotion of inverse relational thinking in younger children.

7.3 Limitations of the Research and Future Directions

It is not possible to explain or pursue every finding that occurs in the course of a thesis. However it is important to highlight which findings provide the most promising avenues for future research beyond the scope of the current thesis. The first limitation of the thesis which could be developed further was that of why children were able to solve speed and some cost problems involving inverse relations earlier than the direct form of the problem. It would be interesting to explore in greater detail why this is the case and also to look at how early children are able to use inverse relations in these easier inverse relations contexts.

The second limitation of the thesis concerns the age of the children involved in the studies. It was the aim of the thesis to look specifically at children who were on the cusp of the logic of relational thinking about quantities (7–9 years). Chapter 6 (Study 4) showed that for children in this age range the additional requirement of computations makes relational thinking very difficult. Presenting children with computational problems in study 4 eliminated the main of effect of quantity reported in chapters 4 and 5 (studies 2 and 3) for the sample. This result leaves two questions unanswered which could be followed up with further research. The first is the length of time it takes children to progress from understanding the logic of direct and inverse relations to realising that the relations between quantities are related multiplicatively and not additively. The second question is whether children develop a computational understanding of relations sooner for extensive than intensive quantities. Although the current thesis identified the distinction between extensive and intensive quantities, work is still needed on the development of this distinction across ages.

The third limitation of the current thesis which would benefit from further research stems from the fact that the studies were all exploratory and that the aim of the thesis was to examine children's knowledge rather than extend it. A future direction of research in the area of children's development in an educational context could examine whether children's development of the ideas relevant to intensive quantity understanding and relational thinking in general can be promoted through intervention work. Here the three main questions from the thesis would be:

- Can explicit teaching about direct and inverse relations in an extensive quantity context promote relational thinking in intensive quantity contexts?
- Is it possible to use children's different ideas of relational thinking in intensive quantity contexts to promote a general understanding of relational thinking?
- Can increased awareness of direct and inverse relations be used to promote the idea that relations between quantities are related multiplicatively and not additively?

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APPENDICES

Appendix 3.1 Interview protocol for Study 1

Pour

Interview booklet Study 2

Child ID:……………………………………………………

Today's date:………………………………………………

Birthday:……………………………………………………

School ID:…………………………………………………

Class ID:……………………………………………………

We are going to pretend to mix some water together and sugar on the computer.
You can see that one glass has an A under it, and other has a B under it.
So the two glasses have different amounts of water and now we are going to mix in some sugar.
In glass A I want you to put 1 lump of sugar and in glass B put 1 lump of sugar.
Now we must stir them until they dissolve.
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

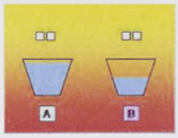
Response and commentary:

W1

So the two glasses have the same amount of water and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve.
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as watery as B?
If no which one will taste watery?
Why will (A/B) taste watery?

SAME/A/B

Response and commentary:



So the two glasses have different amounts of water and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 2 lumps of sugar
Now we must stir them until they dissolve.
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as watery as B?
If no which one will taste watery?
Why will (A/B) taste watery?

SAME/A/B

Response and commentary:

W2

So the two glasses have the same amount of water and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve.
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

Response and commentary:

So the two glasses have different amounts of water and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as watery as B?
If no which one will taste more watery?
Why will (A/B) taste more watery?

SAME/A/B

Response and commentary:

W3

We are going to pretend to mix some lemon juice and sugar together on the computer.. The type of lemon will use, is not the type which you can buy in a bottle, it is the type which you get on trees, do you know what that tastes like? (Talk about what lemon from a tree tastes like let the child taste a bit of lemon).
So in order to make it taste a bit better we are going to mix in some sugar.
Ok I am going to pour some lemon for this big jug here into these two glasss here.
You can see that one glass has an A under it, and other has a B under it. So the two glasses have different amounts of lemon and now we are going to mix in some sugar.
In glass A I want you to put 2 lump of sugar and in glass B put 2 lumps of sugar.
Now we must stir them until they dissolve.
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sour as B?
If no which one will taste sourer?
Why will (A/B) taste sourer?

So the two glasses have different amounts of lemon and now we are going to mix in some sugar
In glass A I want you to put 1 lump of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

Response and commentary:

SAME/A/B

Response and commentary:

L1

So the two glasses have the same amount of lemon and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

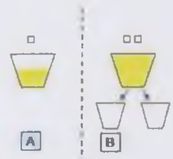
Response and commentary:

L3

So the two glasses have different amounts of lemon and now we are going to mix in some sugar
In glass A I want you to put 1 lump of sugar and in glass B put 2 lumps of sugar
Now we must stir them until they dissolve
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sour as B?
If no which one will taste sourer?
Why will (A/B) taste sourer?

SAME/A/B

Response and commentary:



So the two glasses different amounts of lemon and now we are going to mix in some sugar
In glass A I want you to put 1 lumps of sugar and in glass B put 2 lumps of sugar.
Now we must stir them until they dissolve
I am now going to pour drink A into 2 smaller glasses
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

Response and commentary:

So the two glasses the same amount of lemon and now we are going to mix in some sugar
In glass A I want you to put 2 lumps of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sour as B?
If no which one will taste sourer?
Why will (A/B) taste sourer?

SAME/A/B

Response and commentary:

L2

So the two glasses the same amount of water and now we are going to mix in some sugar
In glass A I want you to put 1 lump of sugar and in glass B put 1 lump of sugar
Now we must stir them until they dissolve
I am now going to pour drink B into 2 smaller glasses
Without tasting the drinks do you think they will taste the same?
If yes then why will A taste as sweet as B?
If no which one will taste sweeter?
Why will (A/B) taste sweeter?

SAME/A/B

Response and commentary:

O3

Appendix 3.2 Letter to head teachers used in Study 1

Dear,

I am writing to ask for permission to visit your school, and work with some of your children. My name is Daniel Bell and I am a PhD student at Oxford Brookes University. I have been involved on various research projects over the past few years with primary aged children in and around Oxford.

I am interested in looking at how children think about maths problems that involve ratio and proportion. I will be analysing how different ways of presenting a problem can make it easier or more difficult for the children to solve the problem. If we know what makes it easier for children to solve a problem, we can find better ways of teaching.

The project is not about differences between children so there will be no comparisons made across children or groups of children. All the information collected will be coded by numbers and no child's name will ever be mentioned in any part of my research.

Each child will be working with me for one session. I will present problems using a laptop computer and other materials. Usually the children enjoy the session very much. The session should take between 20–30 minutes to complete.

If you have any questions you would like to ask before replying, do not hesitate to contact me at Oxford Brookes University on: 01865 483 958.

Thank you for your help

Yours

Daniel Bell

Oxford Brookes University

Appendix 3.3 Letter to parents used in Study 1

Dear Parent

I am writing to ask for permission for your child to take part in a research project about solving maths problems. Your child's nursery has kindly agreed to take part in the project and I now need your permission to work with your child.

My name is Daniel Bell and I am studying for my PhD at Oxford Brookes University. I have worked in various research projects over the past few years with primary aged children in and around Oxford in this project.

I am interested in looking at how children think about maths problems that involve ratio and proportion. I will be analysing how different ways of presenting a problem can make it easier or more difficult for the children to solve the problem. If we know what makes it easier for children to solve a problem, we can find better ways of teaching.

The project is not about differences between children so there will be no comparisons made across children or groups of children. All the information collected will be coded by numbers and your child's name will never be mentioned in any part of my research.

Your child would be working with me for one session. I will present problems using a laptop computer and other materials. Usually the children enjoy the session very much. The session should take between 20–30 minutes to complete.

The study has been approved by the Research Ethics Committee at Oxford Brookes University and that if you have any queries about the conduct of the study, then please contact the Chair on: ethics@brookes.ac.uk

If you **do not** wish your child to participate in the session, please fill in the reply slip below and hand it back to your child's class teacher. If you have any questions you would like to ask before replying, do not hesitate to contact me through the nursery

Thank you for your help
Yours

Daniel Bell
Oxford Brookes University

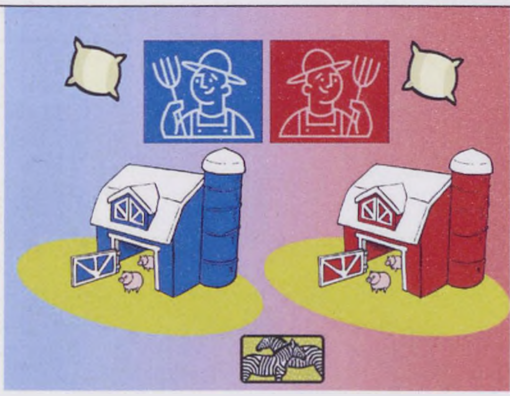
I DO NOT WISH MY CHILD TO TAKE PART IN THE PROJECT

Name of child ....

Class teacher ....

Signed ....

Appendix 4.1 Test items for Study 2

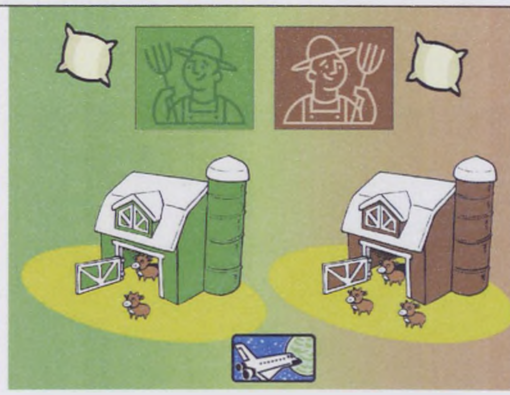
<table>
<tr><th colspan="2">Extensive quantity problems</th></tr>
<tr><th>Direct relations problems</th><th>Inverse relations problems</th></tr>
<tr><td>


Animals controlled, food varied

Farmer Blue and Farmer Red both keep the same number of pigs on their farms.

They both feed their pigs the same amount every day.

Farmer Blue has 20 bags of pig food and farmer Red has 15 bags of pig food.

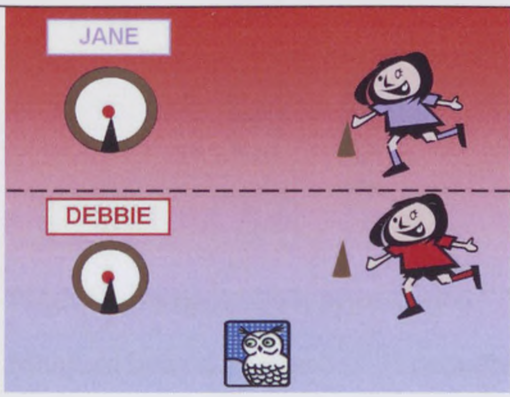
Will one farmer be able to feed his pigs for more days before he has to go shopping for more? Circle yes or no in your booklet. If you circled yes, write the name of the farmer who will be able to feed his pigs for more days.
</td><td>


Animals varied, food controlled

Farmer green and farmer brown both keep cows on their farms.

Farmer green has 10 cows and farmer brown has 15 cows.

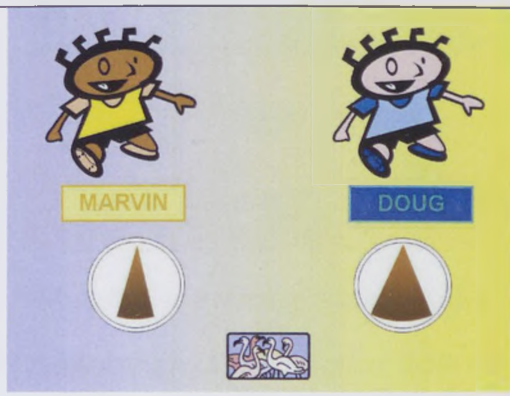
Farmer green and farmer brown have both bought the same number of bags of cattle food each from the market. If they both feed their cows the same amount every day.

Will one farmer be able to feed his cows for more days? Circle yes or no in your booklet. If you circled yes, write the name of the farmer who will be able to feed his pigs for more days.
</td></tr>
<tr><td>


Size of cake varied, size of slices controlled

Jane and Debbie have been playing football in the park.

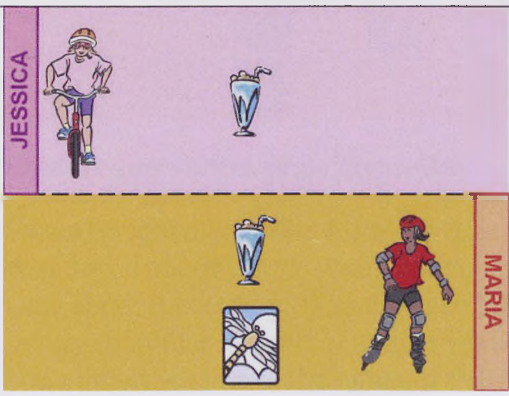
Jane's mum gives her a cake (point) and cuts it into slices this size (point).

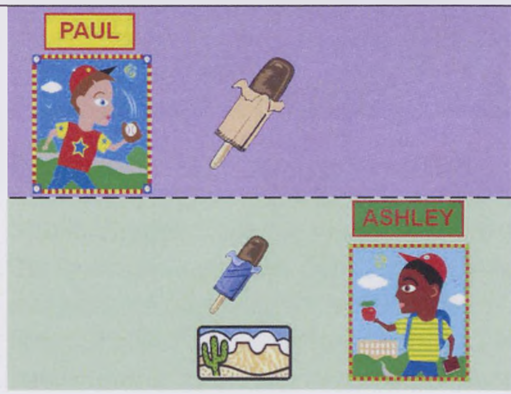
Debbie's mum gives her a cake (point) and cuts it into slices the same size as Jane's
</td><td>


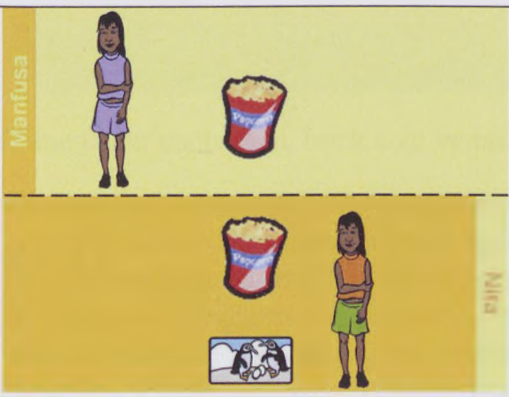
Size of cake controlled, size of slices varied

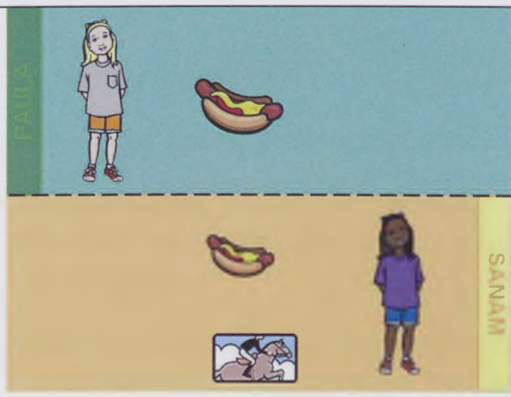
Marvin and Doug both love cake, they each buy the same chocolate cake from the bakers. When Marvin gets home he cuts his cake into slices this size (point).

When Doug gets homes he cuts his cake into slices this size (point).
</td></tr>
</table>

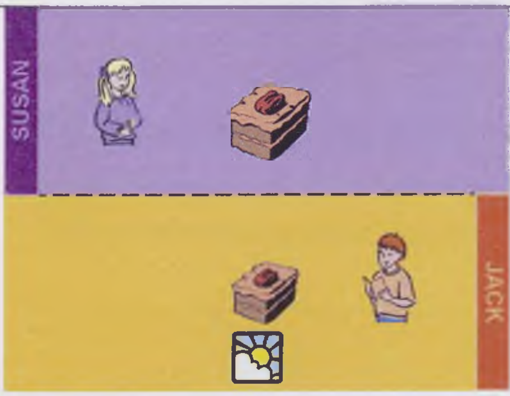
<table>
<tr><td>(point). Who will get the most slices? The children were given three options to circle: Jane; Debbie; they will get the same.</td><td>Who will get the most slices? The children were given three options to circle: Marvin; Doug; they will get the same.</td></tr>
<tr><td colspan="2">Intensive quantity problems</td></tr>
<tr><td>Direct relations problems</td><td>Inverse relations problems</td></tr>
<tr><td>

Product size controlled, price varied
Jessica bought a small milkshake from a shop; it cost 60p. Maria bought a small milkshake in another shop; it cost 65p. Was the milkshake more expensive in one shop than another? Circle yes or no. If you circled yes, write the name of the person whose milkshake was more expensive.</td><td>

Product size varied, price controlled
Paul bought a big ice cream from the corner shop; it cost 85p. Ashley bought a small ice cream in another shop; it cost 85p. Was the ice cream more expensive in one shop than another? Circle yes or no. If you circled yes, write the name of the person whose ice cream was more expensive.</td></tr>
<tr><td>

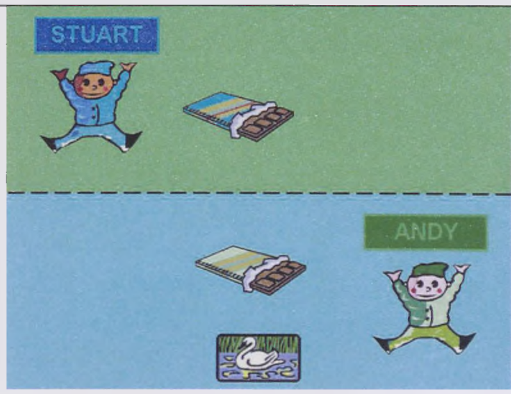
Product size controlled, price varied
Manfusa bought a large box of popcorn from the cinema; it cost 85p. Nita bought a large box of popcorn in another cinema; it cost 75p. Was the popcorn cheaper in one cinema than in the other? Circle yes or no. If you circled yes, write the name of the person whose popcorn was cheaper.</td><td>

Product size varied, price controlled
Paula bought a large hotdog from the cinema; it cost 90p. Sanam bought a small hotdog in another cinema; it cost 90p. Was the hotdog cheaper in one cinema than in the other? Circle yes or no. If you circled yes, write the name of the person whose hotdog was cheaper.</td></tr>
</table>



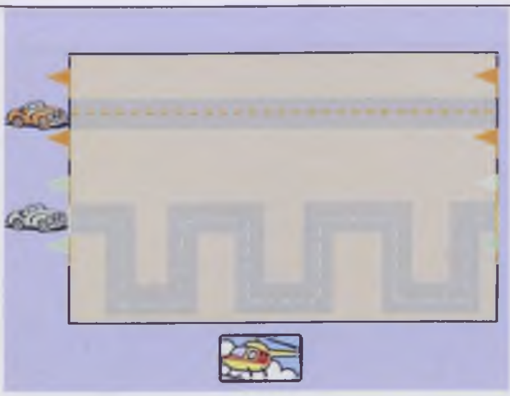
Product size varied, price controlled

Susan bought a big cake from the cake shop; it cost 55p. Jack bought a small cake in another cake shop; it cost 55p. Was the cake better value in one shop than the other? Circle yes or no. If you circled yes, write the name of the person whose cake was better value.



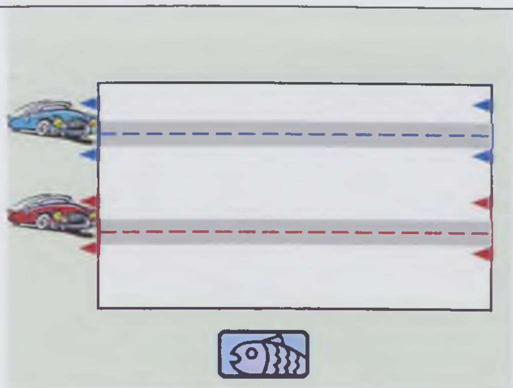
Product size controlled, price varied

Stuart bought a big bar of chocolate from the sweet shop; it cost 30p. Aaron bought a big bar of chocolate in another sweet shop; it cost 35p. Was the chocolate better value in one shop than in the other shop? Circle yes or no. If you circled yes, write the name of the person whose chocolate was better value.



Time taken controlled, track size varied

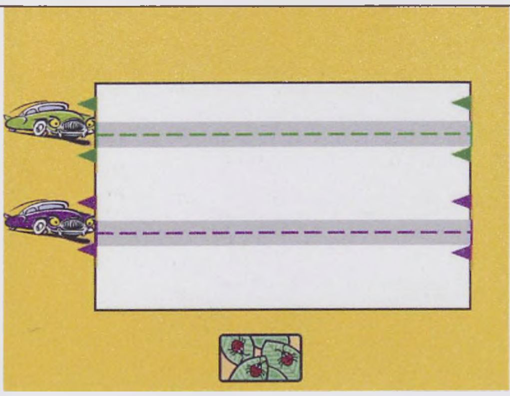
Two cars are on a track. The orange car takes the orange lane and the green car takes the green lane. The orange car reaches the finish line after 82 minutes. The green car reaches the finish line after 82 minutes. Did one car travel quicker than the other? Circle yes or no. If you circled yes, write the colour of the car that travelled more quickly.



Time taken varied, track size controlled

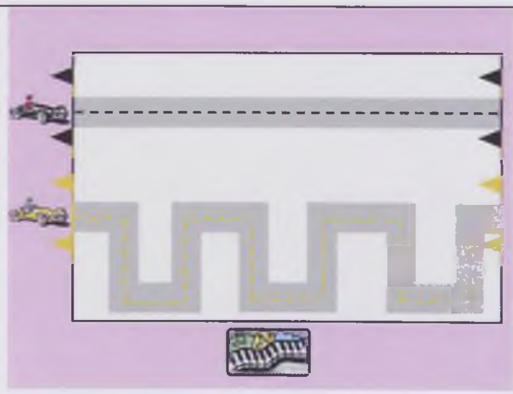
Two cars are on a track. The blue car takes the blue lane and the red car takes the red lane.

The blue car reaches the finish line after 15 minutes. The red car reaches the finish line after 20 minutes. Did one car travel quicker than the other? Circle yes or no. If you circled yes, write the colour of the car that travelled more quickly.



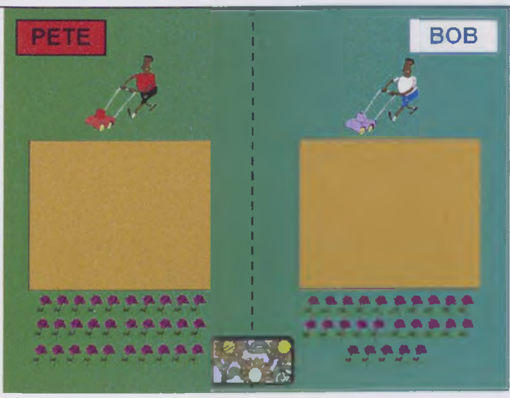
Time taken varied, track size controlled

Two cars are on a track. The green car takes the green lane and the purple car takes the purple lane. The green car reaches the finish line after 55 minutes. The purple car reaches the finish line after 60 minutes. Did one car travel slower than the other? Circle yes or no. If you circled yes, write the colour of the car that travelled more slowly.



Time taken controlled, track size varied

Two cars are on a track. The black car takes the black lane and the gold car takes the gold lane. The black car reaches the finish line after 30 minutes. The gold car reaches the finish line after 30 minutes. Did one car travel slower than the other? Circle yes or no. If you circled yes, write the colour of the car that travelled more slowly.

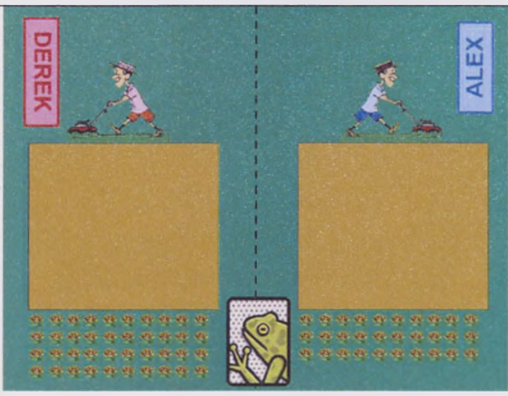


Flowerbed size controlled, number of flowers varied

Pete and Bob are working in their gardens today. First they are cutting the grass. Then they will plant some roses in the flowerbed. Will one flowerbed be more crowded when all the roses have been planted? Circle yes or no. If you circled yes, write the name of the person whose flowerbed will be more crowded.

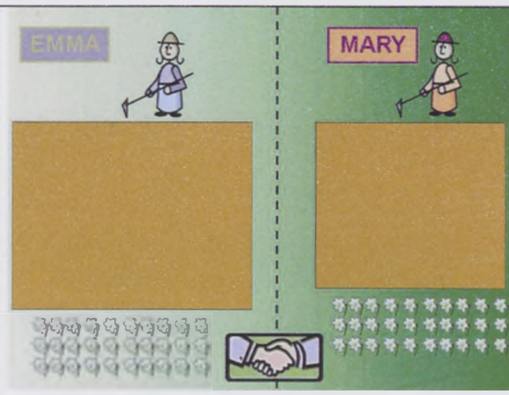
Flowerbed size varied, number of flowers controlled

Rosie and Jim are going to help their parents plant some flowers in the flowerbed. When they have finished will one flowerbed be more crowded than the other? Circle yes or no. If you circled yes, write the name of the person whose flowerbed will be more crowded.



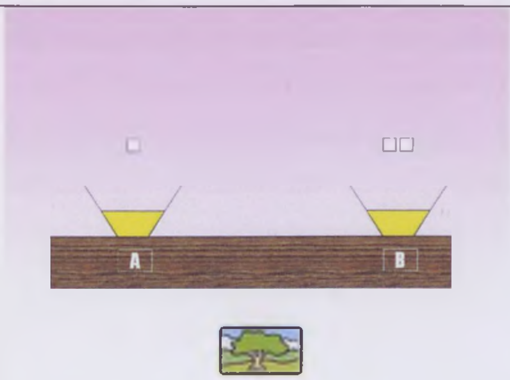
Flowerbed size controlled, number of flowers varied

Derek and Alex both love working in their gardens. The both want to plant some red roses in their new flowerbeds but they have a few other jobs to do first. When they do plant their roses will one of the flowerbeds be less crowded? Circle yes or no. If you circled yes, write the name of the person whose flowerbed will be less crowded.



Flowerbed size varied, number of flowers controlled

Emma and Mary are enjoying a Sunday afternoon in their gardens, both of them are just about ready to plant their lilies in their flowerbeds. When they have finished will one flowerbed be less crowded than the other? Circle yes or no. If you circled yes, write the name of the person whose flowerbed will be less crowded.



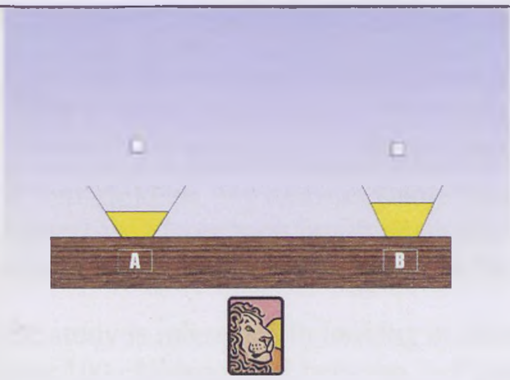
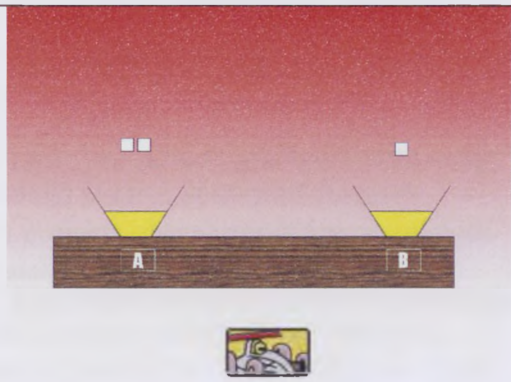
Amount of juice controlled, lumps of sugar varied

The two glasses have the same amount of lemon juice and we are going to mix in some sugar.

In the first cup I put 1 lump of sugar and in the second cup I put 2 lumps of sugar. If you stir them very well, is the juice in this cup (point) going to taste the same as the juice in that cup or will it taste different? Circle same or different. If you think it will taste different, which juice will taste sweeter? Please tick the cup with the sweeter juice.

Amount of juice varied, lumps of sugar controlled

The two glasses have different amounts of lemon juice and we are going to mix in some sugar. In the first cup I put 2 lumps of sugar and in the second cup I put 2 lumps of sugar. If you stir them very well, is the juice in this cup going to taste the same as the juice in that cup or will it taste different? If you think it will taste different, which juice will taste sweeter? Please tick the cup with the sweeter juice.

	
Amount of juice varied, lumps of sugar controlled The two glasses have different amounts of lemon juice and we are going to mix in some sugar. In the first cup I put 1 lump of sugar and in the second cup I put 1 lump of sugar. If you stir them very well, is the juice in this cup going to taste the same as the juice in that cup or will it taste different? If you think they will taste different, which juice will taste sourer? Please tick the cup with the sourer juice.	Amount of juice controlled, lumps of sugar varied The two glasses have the same amount of lemon juice and we are going to mix in some sugar. In the first cup I put 2 lumps of sugar and in the second cup I put 1 lump of sugar. If you stir them very well, is the juice in this cup going to taste the same as the juice in that cup or will it taste different? If you think they will taste different, which juice will taste sourer? Please tick the cup with the sourer juice.

Appendix 4.2 Letter to head teachers used in Study 2

Dear

I am writing to ask for permission to visit your school, and work with some of your children. My name is Daniel Bell and I am a PhD student at Oxford Brookes University. I have been involved on various research projects over the past few years with primary aged children in and around Oxford and Manchester.

The study is interested in looking at the beginning of children's proportional reasoning. Over 100 children aged between 7–9 years, from several local schools are being recruited into the study. The questions I would like to ask to a class of your 7–8 and 8–9 year olds are all child friendly, and will be presented using a laptop computer and projector that I will bring myself. The session should take between 20–30 minutes to complete.

The only aim of this study is to look at the differences between types of questions and not differences between schools or individual children. Children's names will also not be mentioned in any part of my research.

I recognise that you and your staff are very busy, and additional visitors can be inconvenient. If you and your teachers do wish to be part of the study, parental consent will be asked of every child helping me, via a pre-approved letter (see enclosed). Before the session begins I will also speak to the children to ask them if they will help me. Children who don't feel like helping, will be allowed to opt out.

I will also be happy to return to your school at the end of the study if you are interested to talk to you and your staff about my findings, or alternatively I would be happy to send a short report to your school.

Thank you once again for you time and I look forward to speaking with you. I shall follow up this letter with a phone call in about a week's time. If you require any more information please contact my supervisor Professor Terezinha Nunes on: 01865 483 770.

Yours sincerely,

Daniel Bell

Oxford Brookes University

Appendix 4.3 Letter to parents used in Study 2

Dear Parent

I am writing to ask for permission to allow your child to take part in a research project about solving maths problems. My name is Daniel Bell and I am studying for my PhD at Oxford Brookes University. I have been involved on various research projects over the past few years with primary aged children in and around Oxford and Manchester.

I am interested in looking at how children's think about maths problems involving multiplication and division. The children would be working with me for one session in the classroom. Some of the children will also be interviewed about their work. I will be using a laptop computer and projector set up in the classroom that has been enjoyed by children I have worked with previously. The one off session should take between 20–30 minutes to complete.

Your child's school has kindly agreed to take part in the project and I now need your permission to work with your child. The only aim of this study is to look at the differences between types of questions and not differences between schools or individual children. Children's names will also not be mentioned in any part of my research. The children will also be given the opportunity to opt out of the session.

If you do not wish your child to participate in the session, please fill in the reply slip below and hand it back to your child's class teacher within a week. If you have any questions you would like to ask before replying, do not hesitate to contact me on 01865 483 958

Thank you for your help

Daniel Bell

Oxford Brookes University

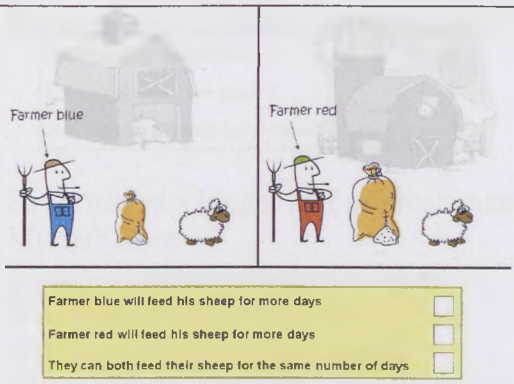
I DO NOT WISH MY CHILD TO TAKE PART IN THE PROJECT

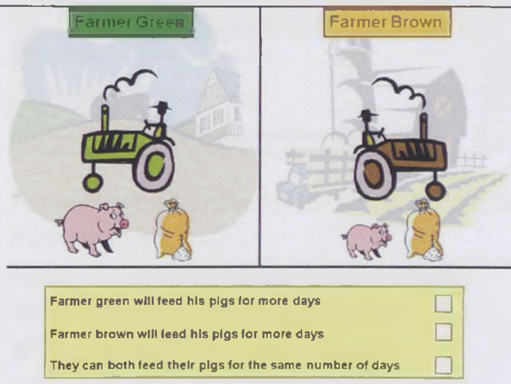
Name of child ....

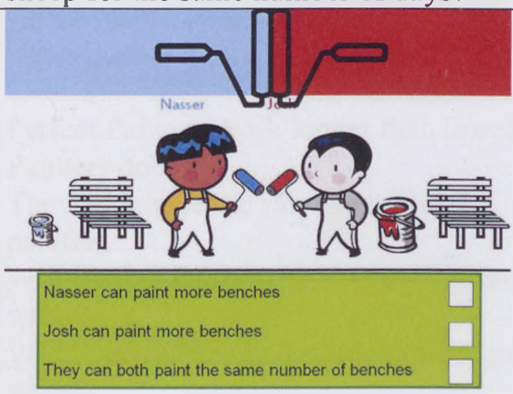
Class teacher ....

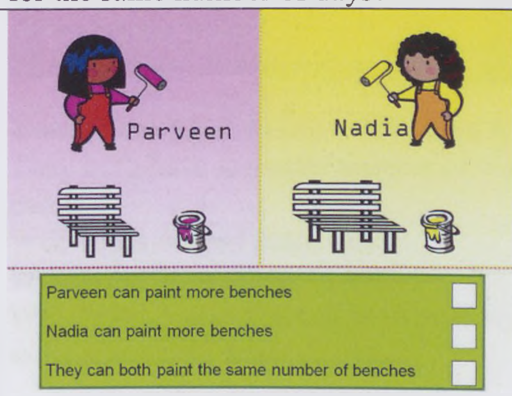
Signed ....

Appendix 5.1 Test items for Study 3

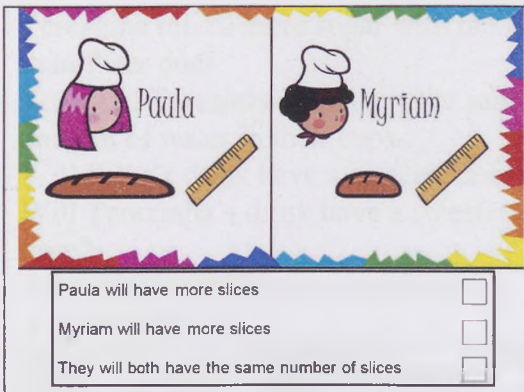
<table>
<tr><th colspan="2">Relational description items</th></tr>
<tr><th colspan="2">Extensive quantities</th></tr>
<tr><th>Direct relations</th><th>Inverse relations</th></tr>
<tr>
<td>


Farmer Red has more sheep food than Farmer Blue.
They both have the same number of sheep.
Will farmer Blue be able to feed his sheep for more days?
Will farmer Red be able to feed his sheep for more days?
Or will they both be able to feed their sheep for the same number of days?
</td>
<td>


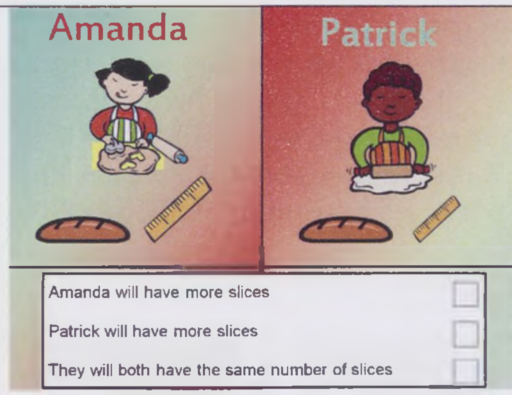
Farmer Green has more pigs than Farmer Brown.
They both have the same amount of pig food.
Will farmer Green be able to feed his pigs for more days?
Will farmer Brown be able to feed his pigs for more days?
Or will they both be able to feed their pigs for the same number of days?
</td>
</tr>
<tr>
<td>


Josh has more paint than Nasser does.
They will both paint benches which are the same size.
Will Nasser be able to paint more benches?
Will Josh be able to paint more benches?
Will Nasser and Josh be able to paint the same number of benches?
</td>
<td>


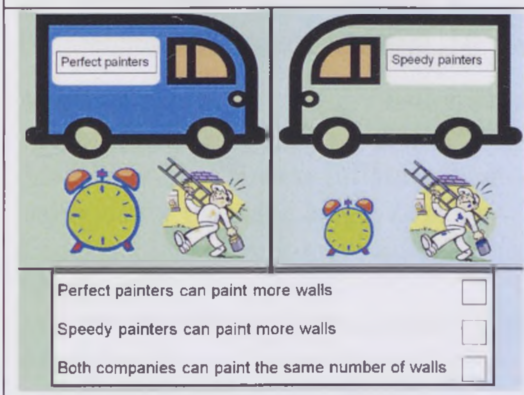
Nadia will paint longer benches than Parveen will.
They will both have the same amount of paint.
Will Nadia be able to paint more benches?
Will Parveen be able to paint more benches?
Will Nadia and Parveen be able to paint the same number of benches?
</td>
</tr>
</table>



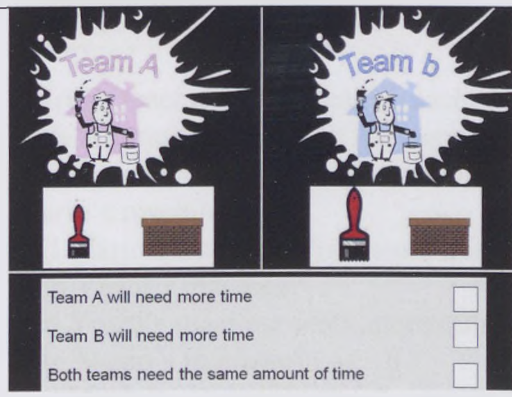
Paula makes a longer loaf of bread than Myriam does.
They both cut the same size slices.
Will Paula have more slices?
Will Myriam have more slices?
Will Paula and Myriam have the same number of slices?



Amanda makes the same sized bread as Patrick does.
Amanda cuts hers into bigger slices.
Will Amanda have more slices?
Will Patrick have more slices?
Will Amanda and Paula have the same number of slices?



Perfect Painters work longer than Speedy Painters does.
They both have the same number of painters.
Will Perfect Painters be able to paint more walls?
Will Speedy Painters be able to paint more walls?
Will Perfect painters be able to paint the same number of walls as Speedy Painters?

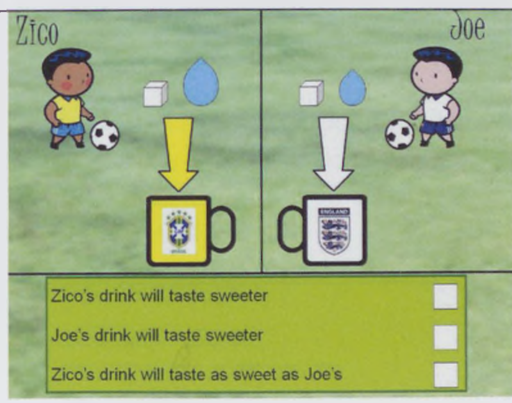


Team B has more painters than team A.
They both have the same number of walls to paint.
Will Team A need more time?
Will Team B need more time?
Will Team A and Team B both need the same amount of time?

Intensive quantity items

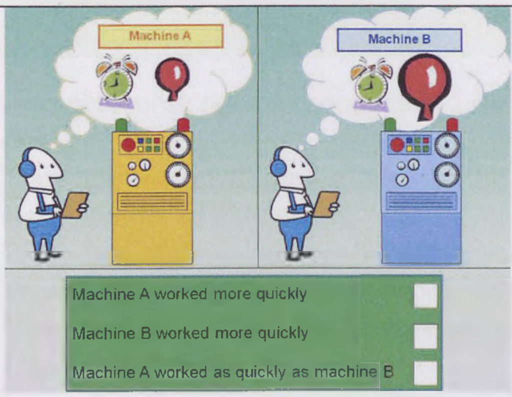
Direct relations

Inverse relations

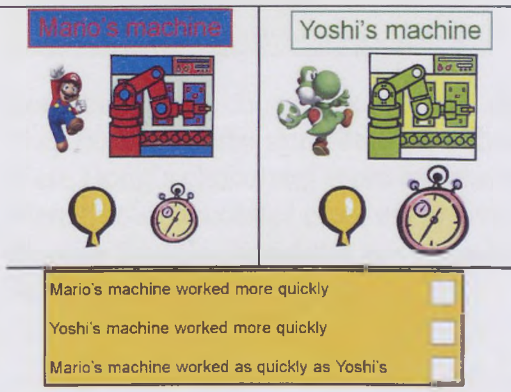


Terezinha mixes more sugar with the water than Peter does.
Peter and Terezinha both have the same amount of water in their cups.
Will Peter's drink have a sweeter taste?
Will Terezinha's drink have a sweeter taste?
Will Peter's drink taste as sweet as Terezinha's?

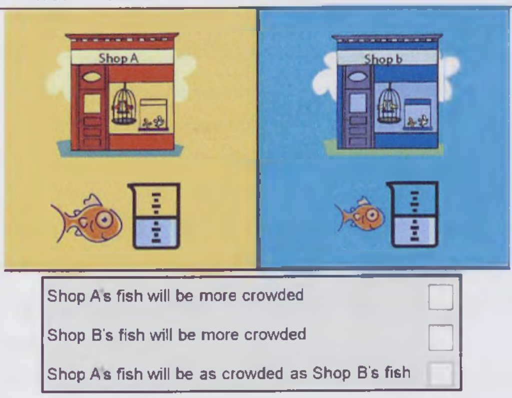
Zico has more water in his cup than Joe does.
Zico and Joe both mix the same amount of sugar into their drinks.
Will Zico's drink have a sweeter taste?
Will Joe's drink have a sweeter taste?
Will Zico's drink taste as sweet as Joe's?



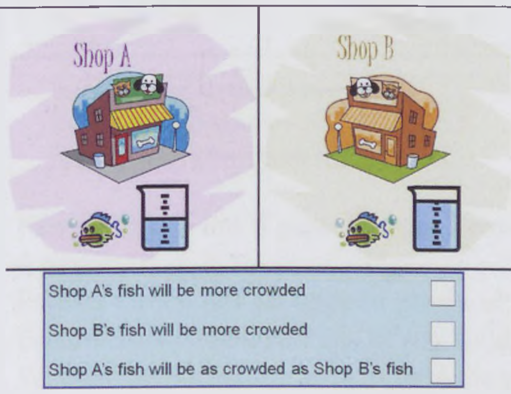
Machine B inflates a bigger balloon than machine A does.
Machine A and B were inflating their balloons for the same length of time.
Did Machine A work more quickly than Machine B?
Did machine B work more quickly than Machine A?
Did Machine A work as quickly as Machine B?



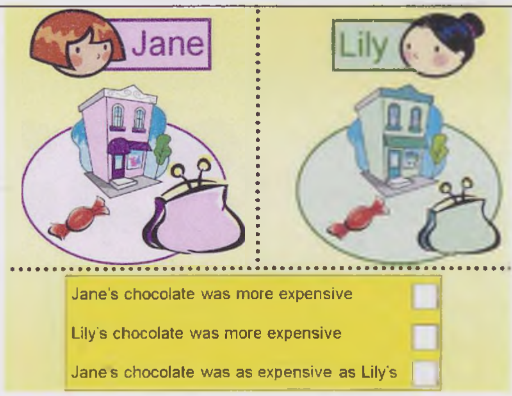
Mario's machine and Yoshi's machine are both inflating a balloon to the same size.
Yoshi's machine takes more time than Mario's machine.
Did Mario's machine work more quickly than Yoshi's machine?
Did Yoshi's machine work more quickly than Mario's machine?
Did Mario's machine work as quickly as Yoshi's machine?

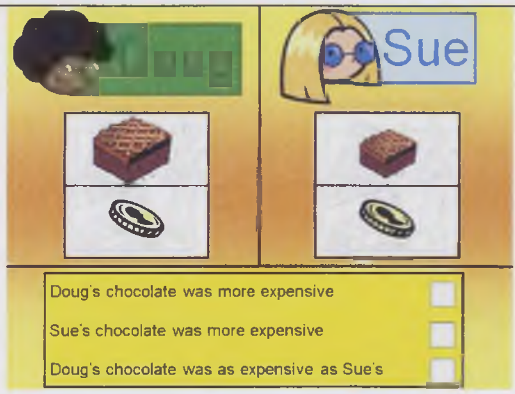


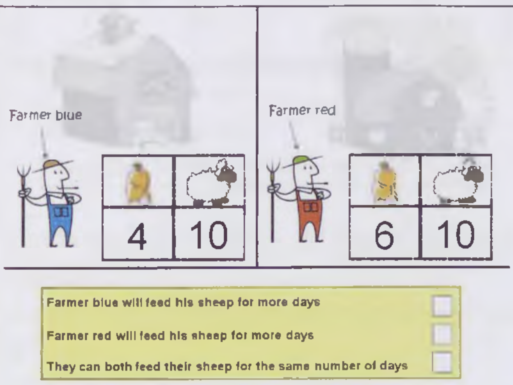
Shop A has more fish than Shop B does
Their tanks hold the same amount of water.
Will the fish in Shop A be more crowded?
Will the fish in Shop B be more crowded?
Will the fish in Shop A be as crowded as the fish in Shop B?



Shop B's tank has more water in it than Shop A's Tank.
Shop A and Shop B both keep the same number of fish.
Will the fish in Shop A be more crowded?
Will the fish in Shop B be more crowded?
Will the fish in Shop A be as crowded as the fish in Shop B?

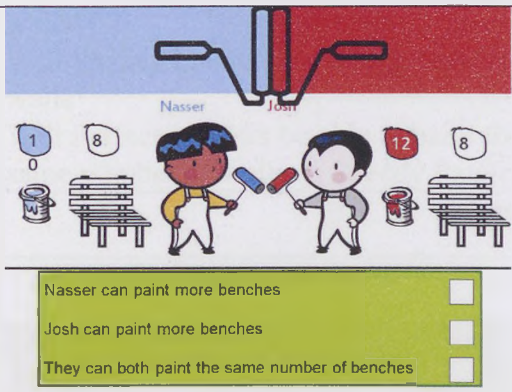
<table>
<tr>
<td>

Jane gets more pocket money than Lily does.
They both spend all their money on the same amount of chocolates.
Were Jane's chocolates more expensive?
Were Lily's chocolates more expensive?
Or were Jane's chocolates as expensive as Lily's?</td>
<td>

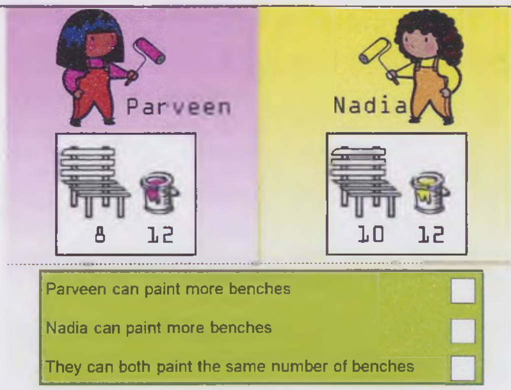
Doug buys more chocolates than Sue does.
They both spent the same amount of money.
Were Doug's chocolates more expensive?
Were Sue's chocolates more expensive?
Or were Doug's chocolates as expensive as Sue's?</td>
</tr>
<tr><td colspan="2">Quantitative description items</td></tr>
<tr><td colspan="2">Extensive quantities</td></tr>
<tr><td>Direct relations</td><td>Inverse relations</td></tr>
<tr>
<td>

Farmer Blue has 4 bags of sheep food
Farmer Red has 6 bags of sheep food.
Farmer Blue has 10 sheep and Farmer Red has 10 sheep.
Will farmer Blue be able to feed his sheep for more days?
Will farmer Red be able to feed his sheep for more days?
Or will they both be able to feed their sheep for the same number of days?</td>
<td>

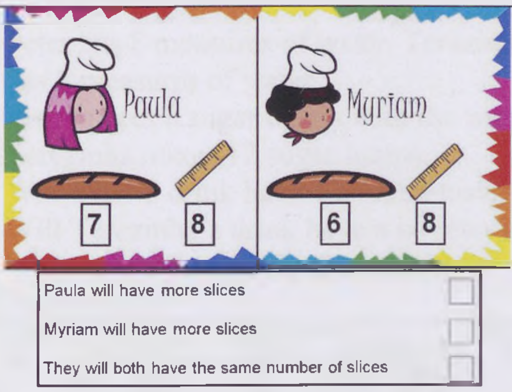
Farmer Green has 6 pigs, Farmer Brown has 4 pigs.
Farmer Green has 10 bags of pig food;
Farmer Brown has 10 bags of pig food.
Will farmer Green be able to feed his pigs for more days?
Will farmer Brown be able to feed his pigs for more days?
Or will they both be able to feed their pigs for the same number of days?</td>
</tr>
</table>



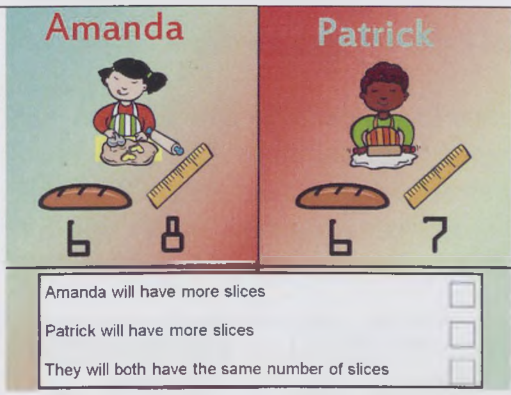
Nasser has 10 tins of paint. Josh has 12 tins of paint.
Nasser will paint benches 8m long, Josh will paint benches 8m long.
Will Nasser be able to paint more benches?
Will Josh be able to paint more benches?
Will Nasser and Josh be able to paint the same number of benches?



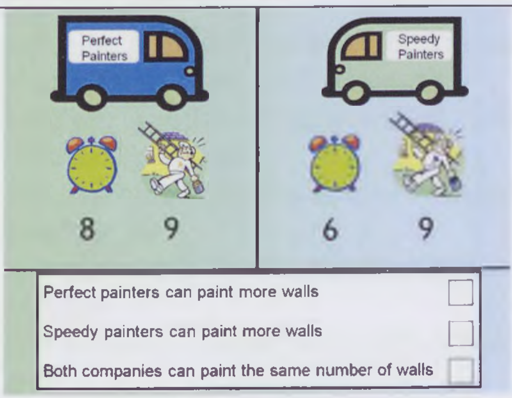
Parveen will paint benches 8m long, Nadia will paint benches 10m long.
Parveen has 12 cans of paint, Nadia has 12 cans of paint.
Will Nadia be able to paint more benches?
Will Parveen be able to paint more benches?
Will Nadia and Parveen be able to paint the same number of benches?



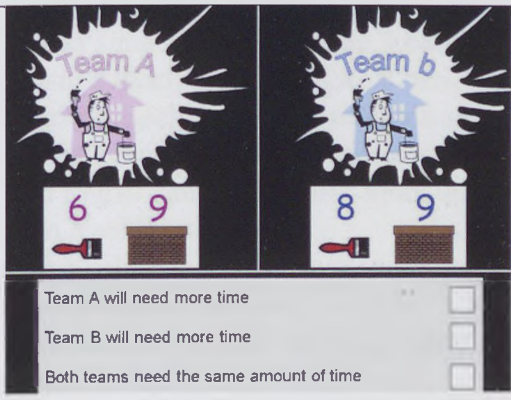
Paula makes 7 loaves, Myriam makes 6 loaves.
Paula cuts 8cm slices. Myriam cuts 8cm slices
Will Paula have more slices?
Will Myriam have more slices?
Will Paula and Myriam have the same number of slices?



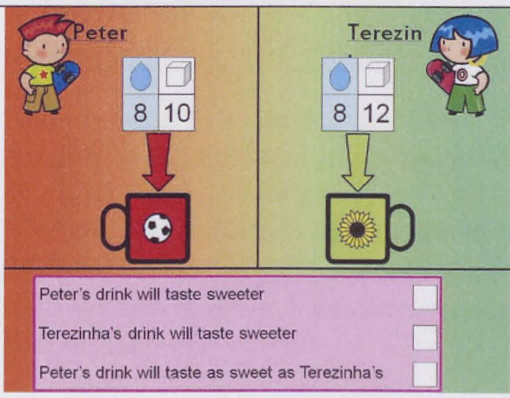
Amanda makes 6 baguettes, Patrick makes 6 baguettes.
Amanda cuts hers into 8cm slices, Patrick cuts his into 7cm slices.
Will Amanda have more slices?
Will Patrick have more slices?
Will Amanda and Paula have the same number of slices?

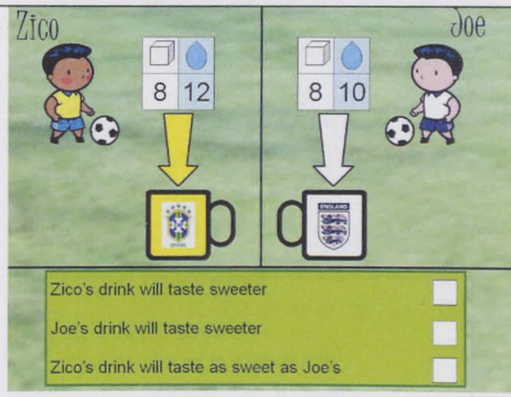


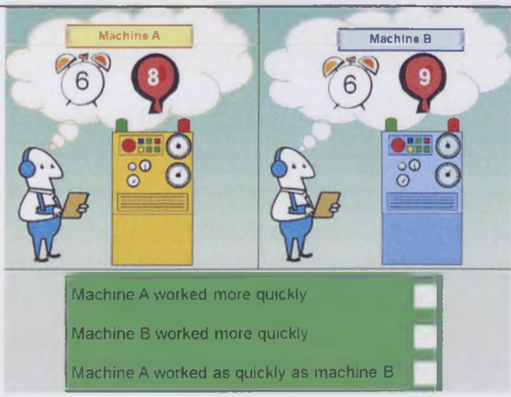
Perfect Painters work for 8 hours and Speedy Painters work for 6 hours.
Perfect Paints has 9 people working, Speedy Painters has 9 people working.
Will Perfect Painters be able to paint more

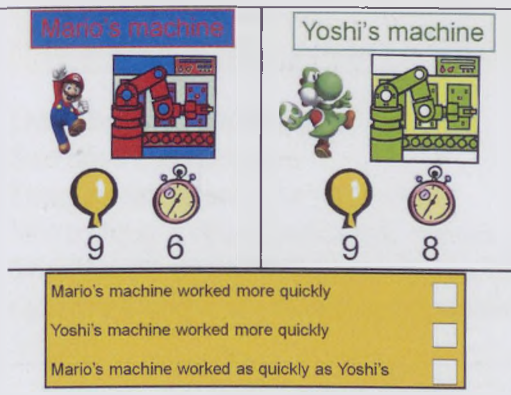


Team A has 6 painters team B has 8 painters.
Team A will paint 9 walls, B will paint 9 walls.
Will Team A need more time?

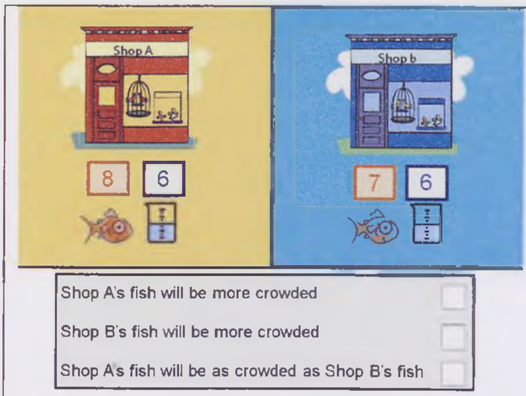
<table>
<tr><td>walls?
Will Speedy Painters be able to paint more walls?
Will Perfect painters be able to paint the same number of walls as Speedy Painters?</td><td>Will Team B need more walls time?
Will Team A and Team B need the same amount of time?</td></tr>
<tr><td colspan="2">Intensive quantity items</td></tr>
<tr><td>Direct relations</td><td>Inverse relations</td></tr>
<tr><td>


Peter has 8 measures of water. Terezinha has 8 measures of water.
Peter mixes 8 sugar lumps with the water, Terezinha mixes 12 sugar lumps.
Will Peter's drink have a sweeter taste?
Will Terezinha's drink have a sweeter taste?
Will Peter's drink taste as sweet as Terezinha's?</td><td>


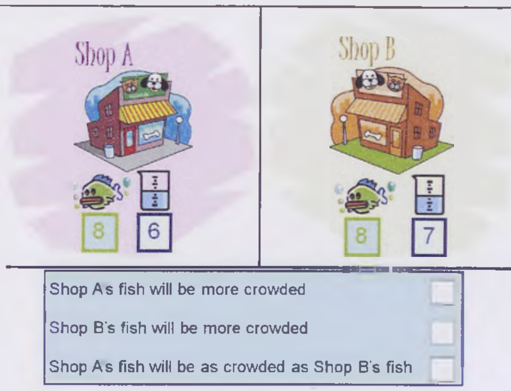
Zico has 8 lumps of sugar. Joe has 8 lumps of sugar.
Zico mixes it with 12 measures of water;
Joe mixes it with 10 measures of water.
Will Zico's drink have a sweeter taste?
Will Joe's drink have a sweeter taste?
Will Zico's drink taste as sweet as Joe's?</td></tr>
<tr><td>


Machine A inflates balloons for 6 minutes.
B inflates balloons for 6 minutes.
Machine A inflates 8 balloons, machine B inflates 9 balloons.
Did Machine A work more quickly than Machine B?
Did machine B work more quickly than Machine A?
Did Machine A work as quickly as Machine B?</td><td>


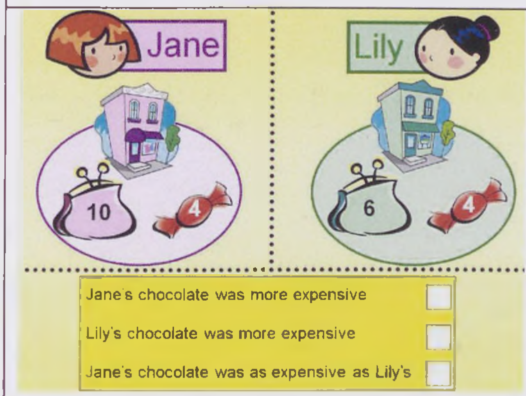
Mario's machine will inflate 9 balloons.
Yoshi's machine will inflate 9 balloons.
Mario's machine takes 6 minutes and Yoshi's machine takes 8 minutes.
Did Mario's machine work more quickly than Yoshi's machine?
Did Yoshi's machine work more quickly than Mario's machine?
Did Mario's machine work as quickly as Yoshi's machine?</td></tr>
</table>



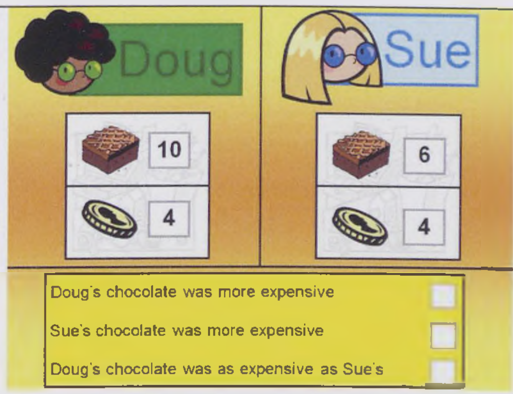
Shop A has 8 fish. Shop B has 7 fish.
Shop A's tank holds 6 litres of water, Shop B's hold 6 litres of water.
Will the fish in Shop A be more crowded?
Will the fish in Shop B be more crowded?
Will the fish in Shop A be as crowded as the fish in Shop B?



Shop A has 8 fish and shop B has 8 fish.
Shop A's tank holds 6 litres of water, shop B's holds 7 litres of water.
Will the fish in Shop A be more crowded?
Will the fish in Shop B be more crowded?
Will the fish in Shop A be as crowded as the fish in Shop B?



Jane has 10p, Lily has 6p.
Jane buys 4 chocolates using all her money
Lily buys 4 chocolates using all her money.
Were Jane's chocolates more expensive?
Were Lily's chocolates more expensive?
Or were Jane's chocolates as expensive as Lily's?



Doug buys 10 chocolates.
Sue buys 6 chocolates.
Doug spent 4p and Sue spent 4p.
Were Doug's chocolates more expensive?
Were Sue's chocolates more expensive?
Or were Doug's chocolates as expensive as Sue's?

Appendix 5.2 Control items for Study 3

Name: ……………………………………………………………………………

Date of birth: ……………………………………………………………………

School:

Year: ………………………………………………………………………………

Today's date:

Gender: boy/girl ID: ………………

73 + 22 = 95 74 + 22 =	19 x 8 = 152 152 ÷ 8 =	17 x 9 = 153 170 x 9 =	98 – 45 = 53 980 – 45 =
192 ÷ 12 = 16 192 ÷ 6 =	276 ÷ 12 = 23 276 ÷ 23 =	98 – 45 = 53 45 – 98 =	252 ÷ 21 = 12 21 ÷ 252 =
59 – 24 = 35 60 – 24 =	112 ÷ 8 = 14 224 ÷ 8 =	73 – 22 = 51 95 – 22 =	68 ÷ 4 = 17 680 ÷ 4 =
24 x 9 = 216 25 x 9 =	48 + 53 = 101 480 + 53 =	144 ÷ 16 = 9 144 ÷ 9 =	92 – 28 = 64 28 + 64 =
247 ÷ 19 = 13 19 ÷ 247 =	91 – 67 = 24 67 – 91 =	16 x 8 = 128 17 x 8 =	29 + 37 = 66 66 – 29 =

Page 1

72 ÷ 3 = 24 72 ÷ 6 =	67 + 24 = 91 24 + 67 =	19 x 13 = 247 13 x 19 =	171 ÷ 9 = 19 171 ÷ 19 =
180 ÷ 12 = 15 360 ÷ 12 =	252 ÷ 21 = 12 273 ÷ 21 =	67 + 24 = 91 670 + 24 =	128 ÷ 16 = 8 144 ÷ 16 =
73 – 22 = 51 74 – 22 =	22 x 5 = 110 110 ÷ 22 =	46 – 33 = 13 47 – 33 =	252 ÷ 21 = 12 253 ÷ 21 =
21 x 12 = 252 22 x 12 =	59 – 24 = 35 83 – 24 =	63 – 36 = 27 36 – 63 =	21 x 12 = 252 12 x 21 =
114 ÷ 19 = 6 1140 ÷ 19 =	72 ÷ 3 = 24 144 ÷ 3 =	198 ÷ 11 = 18 11 x 18 =	112 ÷ 8 = 14 112 ÷ 4 =

Page 2

27 + 36 = 63 36 + 27 =	22 x 6 = 132 132 ÷ 6 =	53 – 18 = 35 53 – 35 =	126 ÷ 9 = 14 1260 ÷ 9 =
45 + 53 = 98 53 + 45 =	27 + 36 = 63 270 + 36 =	26 + 71 = 97 97 – 71 =	135 ÷ 9 = 15 9 x 15 =
19 x 4 = 76 190 x 4 =	52 + 33 = 85 85 – 33 =	92 – 28 = 64 92 – 64 =	28 x 8 = 224 224 ÷ 28 =
63 – 36 = 27 630 – 36 =	53 – 18 = 35 18 + 35 =	47 + 21 = 68 68 – 21 =	14 x 17 = 238 17 x 14 =
238 ÷ 14 = 17 14 ÷ 238 =	54 + 18 = 72 72 – 54 =	312 ÷ 12 = 26 12 x 26 =	91 – 67 = 24 910 – 67 =

Page 3

216 ÷ 24 = 9 217 ÷ 24 =	28 x 5 = 140 140 ÷ 5 =	33 + 46 = 79 34 + 46 =	46 – 33 = 13 79 – 33 =
128 ÷ 16 = 8 129 ÷ 16 =	216 ÷ 24 = 9 240 ÷ 24 =	33 + 24 = 57 57 – 33 =	94 – 38 = 56 38 + 56 =
19 x 6 = 114 114 ÷ 19 =	94 – 38 = 56 94 – 56 =	14 x 6 = 84 140 x 6 =	24 + 59 = 83 25 + 59 =

Page 4

Appendix 5.3 Letter to head teachers used in studies 3 and 4

Dear,

I am writing to ask for permission to visit your school, and work with some of your children. My name is Daniel Bell and I am a PhD student at Oxford Brookes University. I have been involved on various research projects over the past few years with primary aged children in and around Oxford.

I am interested in looking at children's proportional reasoning. I intend to work with 180 children from reception to year 6 at several local schools. There will be 20 child friendly questions would like to ask on a one-to-one basis to a class of your year 3's and year 4's, using a laptop computer and other materials that I will bring myself. The session should take between 30–40 minutes per child to complete.

I can give you my greatest assurance that I am interested in the differences between question types and not schools. Your children will not be mentioned in any part of my research. I can also assure you that my only interest is how children think about problems, so they will not be compared to each other in any way whatsoever.

If you can allow me the time to visit your school, parental consent will be asked of every child helping me, via a pre-approved letter (see enclosed), sent home with the children before I am due to visit. Before the session begins I will also speak to the children to ask them if they will help me. Any children who don't feel like helping don't have too.

I recognise that you and your staff are very busy, and if you think my visit would cause too much inconvenience then that is not a problem.

I will also be willing to return to your school at the end of the study if you are interested to talk to you and your staff about my findings, or alternatively I would be happy to send a short report to your school.

Thank you once again for you time and I look forward to speaking with you, and I shall follow up this letter with a phone call in about a week's time. If you have any concerns about any ethical issues regarding then study then please contact the Chair of the University Research Ethics Committee on **ethics@brookes.ac.uk**.

Yours sincerely,

Daniel Bell M.A. (Child Development).

Appendix 5.4 Letter to parents used in studies 3 and 4
Dear Parent

I am writing to ask for permission to allow your child to take part in a research project about solving maths problems. My name is Daniel Bell and I am studying for my PhD at Oxford Brookes University. I have been involved on various research projects over the past few years with primary aged children in and around Oxford.

I am interested in looking at how children's think about mathematics problems that involve ratios. Your child would be working with me for one session. I will present problems using a laptop computer and other materials. The one off session should take between 30–40 minutes to complete.

Your child's school has kindly agreed to take part in the project and I now need your permission to work with your child. I can give you my greatest assurance that I am interested in the differences between questions and not children so your child will not be compared to any other child in my project. Your child's name will also never be mentioned in any part of my research.

If you wish to allow your child to participate in the session, please fill in the reply slip below and hand it back to your child's class teacher. If you have any questions you would like to ask before replying, do not hesitate to contact me through the school. If you have any concerns about the way in which the study is being conducted, then please contact the Chair of the University Research Ethics Committee on **ethics@brookes.ac.uk**.

Thank you for your help

Yours

Daniel Bell

Oxford Brookes University

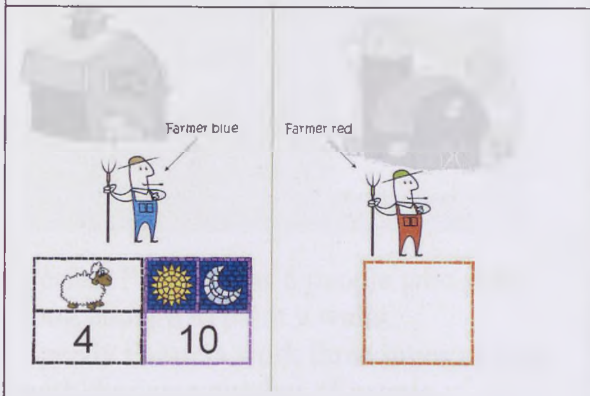
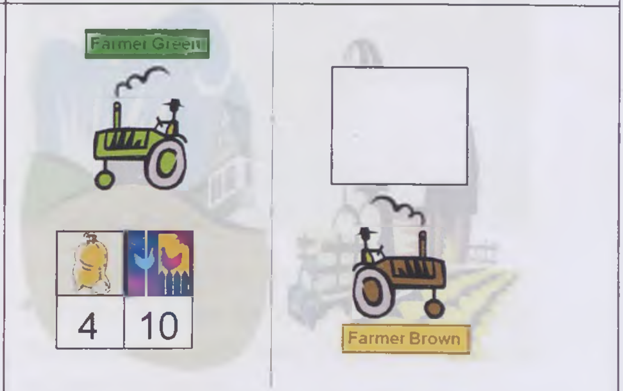
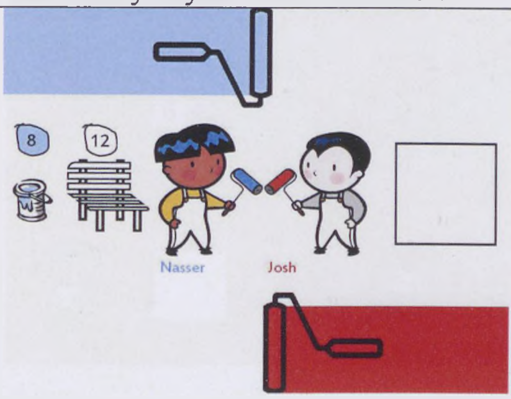
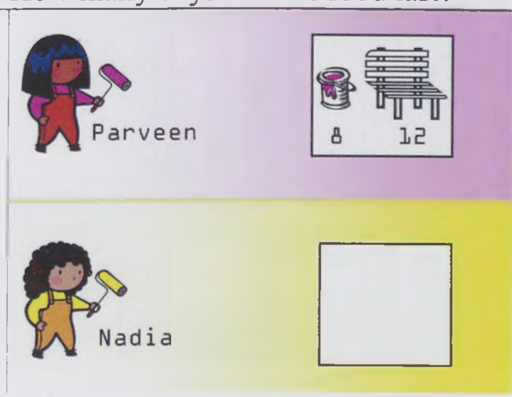
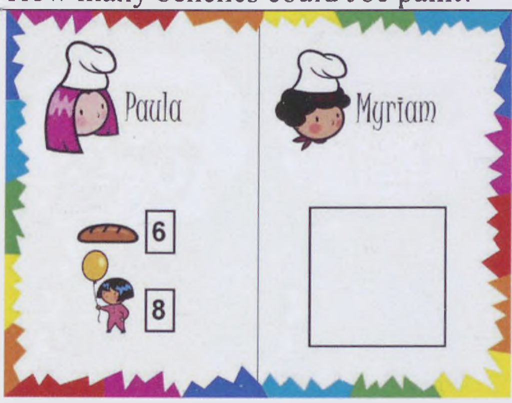
I WOULD LIKE MY CHILD TO TAKE PART IN THE PROJECT

Name of child ....

Class teacher ....

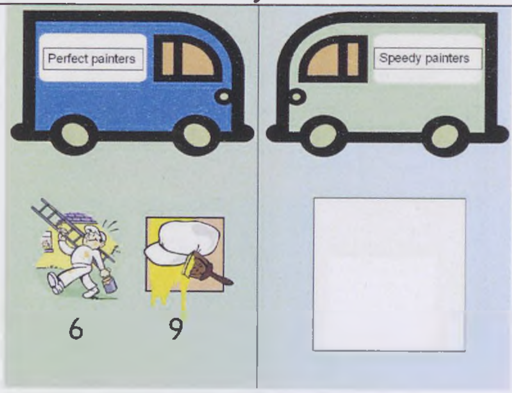
Signed ....

Appendix 6.1 Test items for Study 4

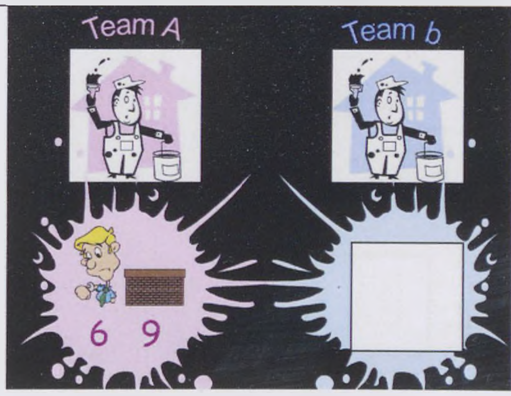
Computational items	
Extensive quantities	
Direct relations	Inverse relations
	
Farmer Blue has 4 sheep. He bought enough food to feed them for 10 days. Farmer Red bought twice as much food and has the same number of sheep. How many days will his food last?	Farmer Green has 4 bags of pig food. It will feed his pigs for 10 days. Farmer Brown has twice as many pigs and the same amount of food. How many days will his food last?
	
Nasser has 8 tins of paint big enough to paint 12 benches. Josh has the same number of tins and his tins are four times bigger. How many benches could Joe paint?	Parveen has 8 tins of paint big enough to paint 12 benches. Nadia has the same number of tins but the benches she will paint are four times bigger. How many benches could Nadia paint?
	
Paula makes 6 loaves of bread which are long enough to share among 8 children. Myriam wants to share bread among twice the number of children and bakes loaves	Amanda makes 6 loaves of bread long enough to share among 8 children. Patrick wants to share bread among the same number of children and bakes loaves which

which are the same length.
How loaves will Myriam need to bake?

are twice as long.
How loaves will Patrick need to bake?



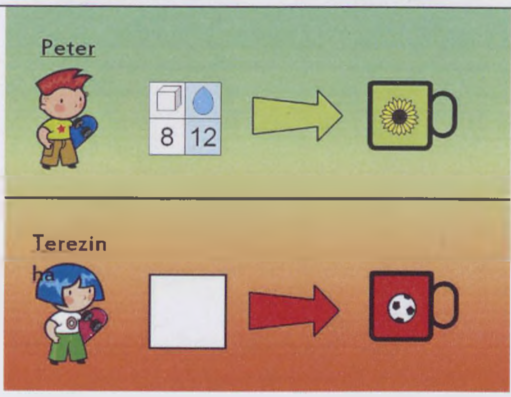
Perfect Painters has 6 people who work long enough to paint 9 walls.
Speedy Painters work three times as long with the same number of people.
How many walls could they paint?



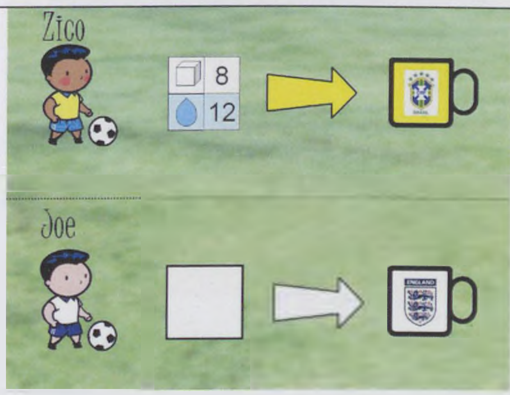
Team A takes 6 hours to paint 9 walls.
In team B there are three times as many painters and they need to paint the same number of walls.
How many hours will it take?

Intensive quantity items

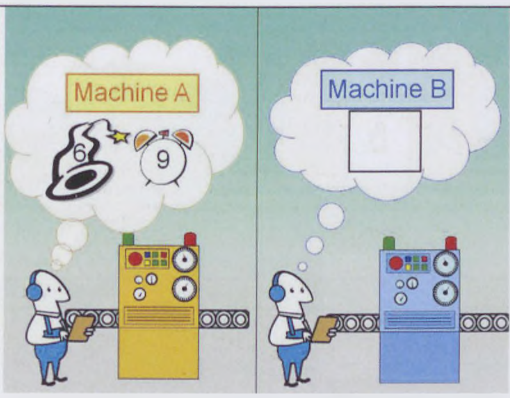
Direct relations	Inverse relations



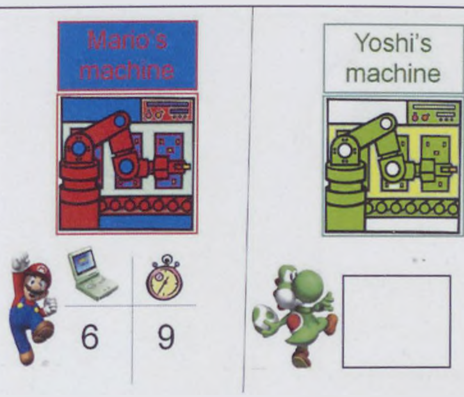
Peter's drink has 8 lumps of sugar mixed with 12 measures of water.
Terezinha's drink has the same amount of water but tastes four times sweeter.
How many lumps of sugar are in Terezinha's drink?



Zico's drink has 8 lumps of sugar mixed with 12 measures of water.
Joe's drink has the same amount of sugar but tastes four times sweeter.
How many measures of water are in Joe's drink?



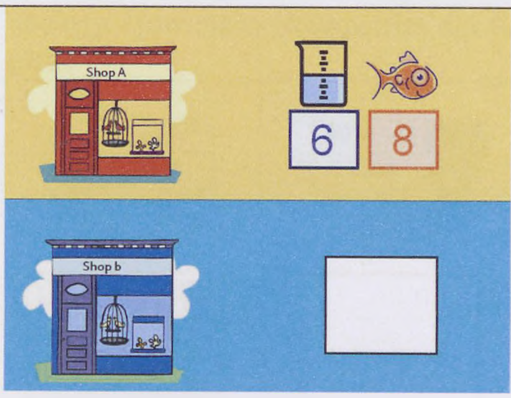
In the factory machine A makes 6 hats in 9 minutes.
Machine B is three times quicker and runs for the same length of time.



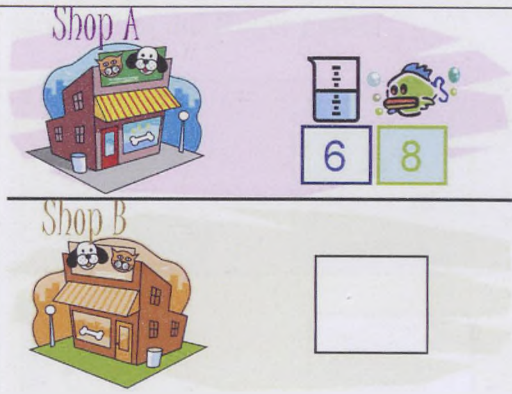
Mario's machine makes 6 gameboys in 9 minutes.
Yoshi's Machine is three times quicker and makes the same number of gameboys.

How many hats will machine B make?

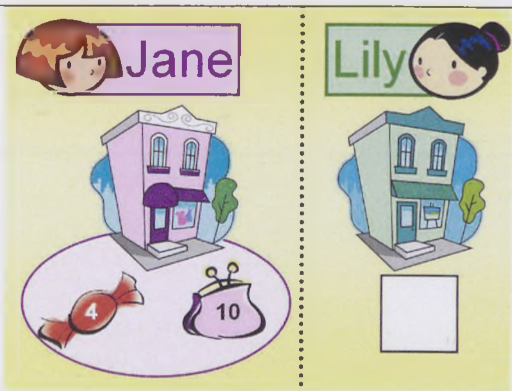
How many minutes did Yoshi's machine take?



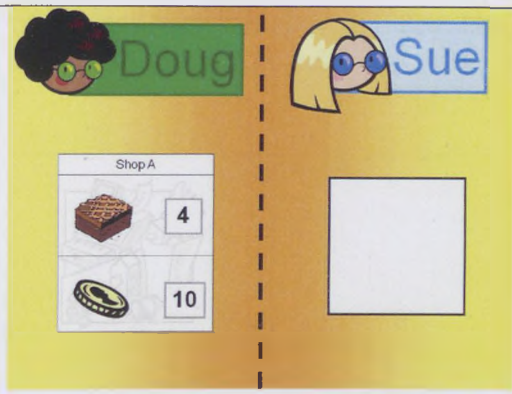
Shop A's fish tank holds 6 litres of water and 8 fish are kept in it.
Shop B's tank holds the same amount of water but the fish are twice as crowded.
How many fish does shop B have?



Shop A has a fish tank that holds 6 litres of water and 8 fish are kept in it.
Shop B has the same number of fish but the tank is twice as crowded.
How many litres of water are in Shop B's tank?

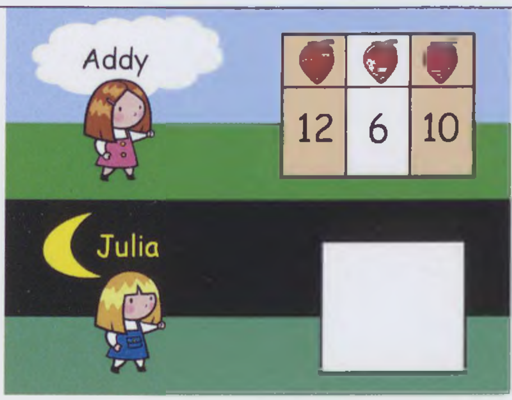


Jane buys 4 chocolates for 10p.
Lily goes to a shop where chocolate is twice as expensive and buys the same amount of chocolate as Jane.
How much money will Lily pay?

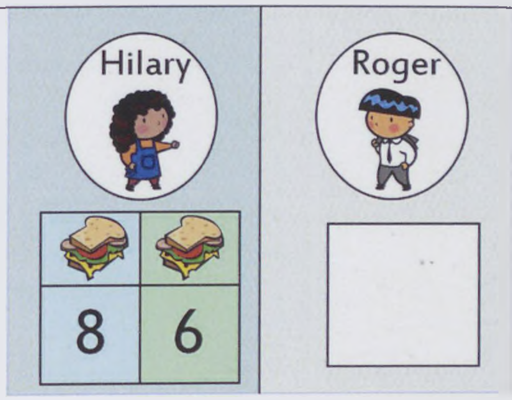


Doug buys 4 chocolates for 10p.
Sue goes to a shop where chocolate is twice as expensive and spends the same amount of money as Doug.
How many chocolates will Sue get?

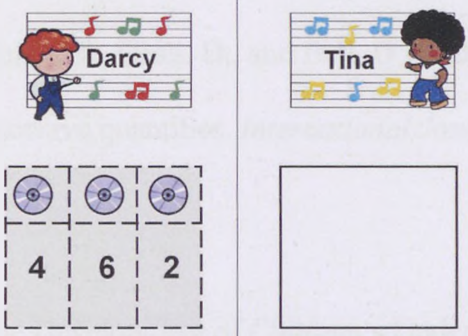
Additive distracter items

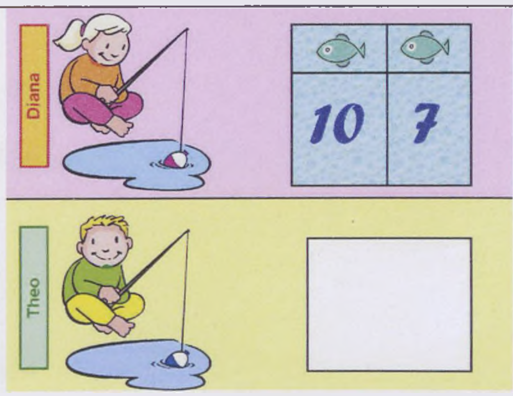


Addy picks strawberries 3 times. On the first day she picks 12 strawberries, on the second day 6 strawberries and on the third



Hilary makes sandwiches twice. On the first day she makes 8 sandwiches and on the second day she makes 6 sandwiches. Roger

<table>
<tr>
<td>day 10 strawberries. Julia picks the same amount of strawberries as Addy but all in one day. How many strawberries did Julia pick?</td>
<td>makes the same amount of sandwiches as Hilary but all in one day. How many sandwiches did Roger make?</td>
</tr>
<tr>
<td>

Darcy buys cds 3 times. On the first day he buys 4 cds, on the second 6 cds, and on the third 2 cds.
Tina buys the same amount of cds as Darcy but all in one day. How many cds did Tina buy?</td>
<td>

Diana goes fishing twice, on the first day she catches 10 fish and on the second day she catches 7 fish. Theo catches the same amount of fish as Diana but all in one day. How many fish did Theo catch?</td>
</tr>
</table>