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Robust Adaptive Synchronization of a Class of Uncertain Chaotic Systems with Unknown Time-Delay

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Abstract: In this paper, a robust adaptive control strategy is proposed to synchronize a class of uncertain chaotic systems with unknown time delays. Using Lyapunov theory and Lipschitz conditions in chaotic systems, the necessary adaptation rules for estimating uncertain parameters and unknown time delays are determined. Based on the proposed adaptation rules, an adaptive controller is recommended for the robust synchronization of the aforementioned uncertain systems that prove the robust stability of the proposed control mechanism utilizing the Lyapunov theorem. Finally, to evaluate the proposed robust and adaptive control mechanism, the synchronization of two Jerk chaotic systems with finite non-linear uncertainty and external disturbances as well as unknown fixed and variable time delays are simulated. The simulation results confirm the ability of the proposed control mechanism in robust synchronization of the uncertain chaotic systems as well as to estimate uncertain and unknown parameters.

Keywords: adaptive controller; unknown time-delayed chaotic system; robust synchronization uncertainty; Lyapunov theory; chaotic system; chaos theory

1. Introduction

The dynamic behaviors of a chaotic system looks like a random behavior. But in reality, the behavior of chaotic systems follows a natural order. On the other hand, the dynamic behavior of these chaotic systems is entirely dependent on the initial conditions, so that with the slightest change in these conditions, the behavior of these systems undergoes extreme changes [1]. Due to the properties of chaotic systems (quasi-random dynamic behavior and drastic dependence on initial conditions), it is difficult to control these systems [2]. Accordingly, this challenge has become a fascinating topic for researchers in various fields [3]. One of the significant issues in the field of control of chaotic systems is the synchronization of such systems in the presence of various disturbances and uncertainties so that, by designing a control strategy, the dynamic behavior of the slave system follows the dynamics of the master system [4–6]. This issue was first introduced by Pecora and Carroll [7]. Owing to the highly substantial applications of synchronization such as secure communications, chemical processes, biological systems, and information processing, numerous control approaches have been submitted to synchronize chaotic systems. Among these techniques the active control method [7], adaptive control strategy [8,9], adaptive observer base control strategy [10], sliding mode control method [11], predictive control [12], Linear feedback control [13], back-stepping control [14], robust control method [15], etc. can be mentioned. In many real and engineering systems comprising biological, physical, electrical, chemical, and communication systems, various uncertainties and time delays are unavoidable and create a lot of changes in the dynamic behavior of these systems and increase the complexity of the systems models. In these systems, achieving the desired behavior is hard and, therefore, the issues of stability and control of these systems have turned out to be a crucially important and a considerable topic. Accordingly, researchers have developed a variety of control strategies for synchronizing systems with uncertainty [16–18] and systems with a time delay [19–23]. In [16], fuzzy-neural network control for synchronization of uncertain chaotic systems is presented. In [20], an iterative learning controller to synchronize two non-linear systems with free time delay and couple free is proposed. An adaptive fuzzy controller design for synchronizing a class of uncertain non-linear systems is proposed in [17]. In [22], an adaptive intelligent controller is introduced to control uncertain systems with time delay. An observer-based fuzzy adaptive output feedback controller [21] is designed to control the stochastic system with time delay. In [19], the design of a synchronization controller based on robust adaptive neural networks for a class of time-delay uncertain chaotic systems is yielded. Adaptive synchronization has been proposed for a class of time-delay uncertain chaotic systems via fuzzy fractional-order neural networks [23].

Due to the increasing complexity of time-delayed systems, the synchronization of time-delayed chaotic systems with the application approach in secure telecommunications has also received great attention from researchers in this field. For example, finite-time synchronization of non-identical chaotic systems with multiple time-varying delays and bounded disturbances was investigated in [24]. Chen et al. [25] adopted an improved synchronization to synchronize time-delayed chaotic Lur'e systems using sampled-data control. Also, in [26], Zheng et al. proposed a novel synchronization criterion of chaotic Lur'e systems with time delays using sampled-data control. A control strategy for heterogeneous uncertain chaotic systems with a time delay [27] is proposed in which a robust framework for synchronization error estimation is presented. To illustrate the proposed approach in engineering applications, thishas been exploited for secure communication via multiple heterogeneous chaotic systems. In [28], an observer-based sliding mode control method is proposed to synchronize the time-delayed chaotic neural networks with unknown perturbation. The utilization of the proposed controller for the master and slave chaotic systems with non-identical structure as well as the slave system with unknown disturbance is the advantage of the proposed procedure.

One of the items that can be considered for time-delayed uncertain systems is the uncertainty of the time delay existingin the system. This issue can pose serious challenges to the controller design process for a variety of purposes, including the synchronization process. On the other hand, the uncertainty of the amount of time delay in the system gives rise to an increase in the degree of complexity of the system model, which can be considered asenhancing the level of data security in the field of secure telecommunications. Accordingly, this paper deals with an adaptive control strategy for robust synchronization of uncertain chaotic systems with unknown fixed and variable time delay. Using Lyapunov's direct method and the Lipschitz condition in chaotic systems, the update rules, and the estimation of the unknown parameters are determined to ensure the stability of the proposed controller. Finally, in order to evaluate the performance of the proposed control strategy, the synchronization of two Jerk chaotic systems with unknown fixed and variable time-delay coupled with uncertainty and disturbance are investigated and simulated. The simulation results exhibit the effectiveness of the proposed adaptive control method for robust synchronization due to uncertainty, external distortions, and unknown fixed and variable time-delays. Also, based on the results, the proposed control strategy has been effective in estimating uncertain parameters and unknown fixed and variable delays.

This paper is organized as follows: in Section 2, the basic definitions used in this paper are presented. Adaptive controller design to robust synchronization of fractional order chaotic systems with unknown time-delay, uncertainty and disturbance is givenin Section 3. The synchronization of two Jerk chaotic systems with unknown fixed and variable time delay and uncertainty and disturbance is analyzed and simulated in Section 4, and finally, in Section 5, conclusions based on the stated theories are presented.

2. Preliminaries and Problem Formulation

In this paper, a class of uncertain chaotic systems with unknown time delay is considered so that the general form of demonstrating dynamic equations to its companion form in the format of master and slave systems is defined as follows:

$$\begin{cases} \dot{x}_i = x_{i+1}, \quad 1 \le i \le n-1 \\ \dot{x}_n = f(x(t-\tau_1),t) + \Delta f(x(t),t) + d_1(t), \end{cases} \tag{1}$$

and the slave system is written in the following form:

$$\begin{cases} \dot{y}_i = y_{i+1}, \quad 1 \le i \le n-1 \\ \dot{y}_n = g(y(t-\tau_2),t) + \Delta g(y(t),t) + d_2(t) + u(t). \end{cases} \tag{2}$$

So that $x(t),\ y(t) \in R^n$ denote the dynamic states of the master and slave systems, $f(x(t-\tau_1),t),\ g(y(t-\tau_2),t) \in R$ nonlinear functions with unknown time delays with delays $\tau_1,\ \tau_2$ and $\Delta f(x(t),t),\ \Delta g(x(t),t)$ express nonlinear bounded uncertainties in the slave and master systems. Also, $d_1(t),\ d_2(t)$ indicate the external disturbances applied to the master and slave systems, and $u(t)$ the control law applied to the slave system. The differential equations expressed in the forms correspond to a number of known chaotic systems such as: Van der Pol oscillator systems, Duffing's oscillator, Genesio-Tesi's system, Arneodo's system, etc. [29].

Definition 1. *The master and the slave systems defined in forms (1) and (2) have robust synchronization if for all conditions governing the system, including external disturbance, uncertainties and unknown time-delays, for each initial condition, the following condition must be met:*

$$\lim_{t\to\infty} \|y_i(t) - x_i(t)\| = lim_{t\to\infty}\|e_i(t)\| = 0,\ \ i = 1,\ldots,n. \tag{3}$$

Therefore, $e_i(t)$ describes the synchronization error of the master and the slave systems. Accordingly, the differential equations of the synchronization error dynamics for uncertain master and slave chaotic systems with unknown time-delay express in forms (1) and (2) are defined as:

$$\begin{cases} \dot{e}_i = e_{i+1}, & 1 \le i \le n-1 \\ \dot{e}_n = g(y(t-\tau_2),t) + \Delta g(x(t),t) + d_2(t) - (f(x(t-\tau_1),t) + \Delta f(x(t),t) + d_1(t)) + u(t). \end{cases} \tag{4}$$

Therefore, in this paper, we seek to design a robust adaptive controller to perform robust synchronization of chaotic systems (1) and (2) in the presence of external disturbances, bounded non-linear uncertainties and existing unknown time-delays in accordance with Definition 1. Put another way, the proposed designed controller can operate despite the existing conditions in the master and slave systems in such a way that the state dynamics of the slave system in a finite time is in accordance with the behavior of the dynamics of the master system. Also, the adaptation of state dynamics under the assumed conditions remain stable and robust until the synchronization error tends to zero.

Assumption 1. *Uncertain external disturbances $d_1(t)$, $d_2(t)$ and uncertain bounded non-linear uncertainties $\Delta f(x(t),t)$ and $\Delta g(x(t),t)$ in master and slave systems (1) and (2) satisfy the following conditions:*

$$\|\Delta f(x(t),t)\| \leq \alpha_1 h_1(x),\ \|\Delta g(y(t),t)\| \leq \alpha_2 h_2(y),\ \|d_1(t)\| \leq \gamma_1 \text{ and } \|d_2(t)\| \leq \gamma_2. \tag{5}$$

Thus $\|\cdot\|$ signifies the norm l_1 and α_1, α_2, γ_1, γ_2 are the values of the unknown real positive constant. Also, $h_1(\cdot)$, $h_2(\cdot)$ are generally known functions.

Assumption 2. *The non-linear functions $f(x(t-\tau_1),t)$, $g(y(t-\tau_2),t) \in R$ containing the unknown time delays in the master and slave chaotic systems defined in general forms (1) and (2), for each $x(t)$, $y(t) \in R$, provide the following Lipschitz conditions:*

$$\begin{aligned} \left|f(x(t-\tau_1)) - f(x(t-\hat{\tau}_1))\right| &\leq l_1\|x(t-\tau_1) - x(t-\hat{\tau}_1)\| \leq m_1\left|(t-\tau_1) - (t-\hat{\tau}_1)\right| \\ &= m_1|\tau_1 - \hat{\tau}_1| = m_1\left|\widetilde{\tau}_1\right|, \\ \left|g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2))\right| &\leq l_2\|y(t-\tau_2) - y(t-\hat{\tau}_2)\| \leq m_2\left|(t-\tau_2) - (t-\hat{\tau}_2)\right| \\ &= m_2|\tau_2 - \hat{\tau}_2| = m_2\left|\widetilde{\tau}_2\right|. \end{aligned} \tag{6}$$

where, τ_1, $\tau_2 \in R$ denote unknown time delays, $\hat{\tau}_1$, $\hat{\tau}_2 \in R$ represent unknown time delays estimation, l_1, l_2, m_1 and m_2 are positive and unknown constants.

3. Adaptive Controller Design for Robust Synchronization of Fractional Order Chaotic Systems with Unknown Time Delay, Uncertainty and Disturbance

In this section, the objective is to present a robust adaptive control strategy for robust synchronization of the aforementioned chaotic systems, given the structure of uncertain chaotic systems with unknown time delays defined in forms (1) and (2), in bounded time provided condition in Definition 1 are met. In the following, we prove that in order to fully synchronizethe systems described in forms (1) and (2) in a finite time, at least two controllers $u_0(t)$ and $u(t)$ are required to stabilize the synchronization error dynamic equations, based on which (4) is rewritten as below:

$$\begin{cases} \dot{e}_i = e_{i+1}, \qquad i = 1, 2, \ldots, n-2 \\ \dot{e}_{n-1} = e_n + u_0(t), \\ \dot{e}_n = g(y(t-\tau_2),t) + \Delta g(x(t),t) + d_2(t) - (f(x(t-\tau_1),t) + \Delta f(x(t),t) + d_1(t)) + u(t), \end{cases} \tag{7}$$

Theorem 1. *A necessary condition for the robust synchronization of the chaotic systems described in Forms (1) and (2) in a finite time is that the controller $u_0(t)$ in (7) must be designed in the following form:*

$$u_0(t) = K^T E, \tag{8}$$

where, $E = (e_1, e_2, \ldots, e_n)$ represent the error synchronization dynamics, $K^T = (K_1, K_2, \ldots, K_{n-1}, -1)$ and real values of K_i must selected such that eigenvalues λ_i for $i = 1, 2, \ldots, n-1$ in system (7) satisfy the Hurwitz stability conditions.

Proof. According to the proposed structure based on (7), $n-1$ controlled synchronization error dynamics are presented in the following linear form:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \vdots \\ \dot{e}_{n-2} = e_{n-1}, \\ \dot{e}_{n-1} = e_n + u_0(t). \end{cases} \tag{9}$$

If $u_0(t)$ is designed in the form (8) then the linear time invariant (LTI) dynamic system (9) is converted to the following form:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \vdots \\ \dot{e}_{n-2} = e_{n-1}, \\ \dot{e}_{n-1} = K_1 e_1 + K_2 e_2 + \ldots + K_{n-1} e_{n-1}. \end{cases} \tag{10}$$

It is clear that according to Herwitz's stability theory in LTI systems, if K_i values for $i = 1, 2, \ldots, n-1$, are chosen that all eigenvalues λ_i of the LTI system (10), satisfy conditions $Real(\lambda_i) \leq 0$ then, by designing $u_0(t)$ based on Theorem 1 conditions, all $n-1$ synchronization error dynamics in the system (7) are stable, and this stability is a necessary condition for robust synchronization of chaotic systems with synchronization error in form (7). □

Theorem 2. *Robust synchronization of the master and slave chaotic systems with uncertainty and unknown time delays described in forms (1) and (2) is fulfilled if the controller $u_0(t)$ characterized in form (8) satisfies the condition of Theorem 1 and the adaptive controller $u(t)$ is designed as follows:*

$$u(t) = f(x(t-\hat{\tau}_1)) - g(y(t-\hat{\tau}_2)) - sign(e_n)(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\alpha}_1 h_1(x) + \hat{\alpha}_2 h_2(y)) - k_0 e_n, \tag{11}$$

so that k_0 is a desired positive real value and the adaptation laws in the controller (11) are determined as follows:

$$\begin{gathered} \dot{\widetilde{\alpha}}_1 = -|e_n| h_1(x) - \eta_1 \widetilde{\alpha}_1, \ \ \dot{\widetilde{\alpha}}_2 = -|e_n| h_2(y) - \eta_2 \widetilde{\alpha}_2, \\ \dot{\widetilde{\gamma}}_1 = -|e_n| - \eta_3 \widetilde{\gamma}_1, \ \dot{\widetilde{\gamma}}_2 = -|e_n| - \eta_4 \widetilde{\gamma}_2, \\ \dot{\widetilde{\tau}}_1 = -|e_n| sign(\widetilde{\tau}_1) - \eta_5 \widetilde{\tau}_1, \ \dot{\widetilde{\tau}}_2 = -|e_n| sign(\widetilde{\tau}_2) - \eta_6 \widetilde{\tau}_2, \end{gathered} \tag{12}$$

where η_i for $i = 1, 2, \ldots, 6$ are constant and positive values and $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ \widetilde{\gamma}_1, \ \widetilde{\gamma}_3, \widetilde{\tau}_1, \widetilde{\tau}_2$ denote the estimation errors of uncertain and unknown parameters in the master and slave systems of (1) and (2).

Proof. Based on the direct method of the Lyapunov theory, the following function is proposed:

$$V(e_n, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\tau}_1, \widetilde{\tau}_2) = \frac{1}{2}({e_n}^2 + {\widetilde{\alpha}_1}^2 + {\widetilde{\alpha}_2}^2 + {\widetilde{\gamma}_1}^2 + {\widetilde{\gamma}_2}^2 + m_1 {\widetilde{\tau}_1}^2 + m_2 {\widetilde{\tau}_2}^2). \tag{13}$$

The derivative of the proposed Lyapunov function (13) is yielded in the following form:

$$\dot{V} = e_n \dot{e}_n + \sum_{i=1}^{2} (\widetilde{\alpha}_i \dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i \dot{\widetilde{\gamma}}_i + m_i \widetilde{\tau}_i \dot{\widetilde{\tau}}_i). \tag{14}$$

Based on the error dynamics $\dot{e}_n$ in the differential equations of synchronization error (7) and its substitution in (14), the proposed derivative of the proposed Lyapunov function can be illustrated in the following new form:

$$\begin{aligned}\dot{V} = e_n(g(y(t-\tau_2)) + \Delta g(x(t)) + d_2(t) - f(x(t-\tau_1)) - \Delta f(x(t)) - d_1(t) + u(t)) \\ + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{15}$$

If in Equation (15), $u(t)$ is expressed as follows:

$$u(t) = -g(y(t-\hat{\tau}_2)) + f(x(t-\hat{\tau}_1)) - k_0e_n + \overline{u}(t), \tag{16}$$

where $\hat{\tau}_1$ and $\hat{\tau}_2$ denote estimation of the unknown delays τ_1 and τ_2 in the master and slave systems (1) and (2), and k_0 is an arbitrary positive constant, then (15) is rewritten as follows:

$$\begin{aligned}\dot{V} = e_n(g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2)) + \Delta g(x(t)) + d_2(t) + f(x(t-\hat{\tau}_1)) \\ -f(x(t-\tau_1)) - \Delta f(x(t)) - d_1(t) - k_0e_n + \overline{u}(t)) \\ + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{17}$$

Applying the first norm l_1, it is explicit that (17) can be rewritten as follows:

$$\begin{aligned}\dot{V} \le |e_n|\Big(\big|g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2))\big| + \big|\Delta g(x(t))\big| + \big|d_2(t)\big| \\ +\big|f(x(t-\hat{\tau}_1)) - f(x(t-\tau_1))\big| + \big|\Delta f(x(t))\big| + \big|d_1(t)\big|\Big) - k_0e_n^2 \\ +e_n\overline{u}(t) + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{18}$$

Based on the assumed conditions for disturbances and uncertain uncertainties in the master and slave systems provided in Assumption 1, (18) is revised as follows:

$$\begin{aligned}\dot{V} \le |e_n|\Big(\big|g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2))\big| + \alpha_2h_2(y) + \gamma_2 \\ +\big|f(x(t-\hat{\tau}_1)) - f(x(t-\tau_1))\big| + \alpha_1h_1(x) + \gamma_1\Big) - k_0e_n^2 + e_n\overline{u}(t) \\ + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{19}$$

Due to the fulfillment of Lipschitz conditions (6) by non-linear functions withunknown time delays in the master and slave systems in accordance with Assumption 2, (19) is presented as follows:

$$\begin{aligned}\dot{V} \le |e_n|\Big(m_1\big|\widetilde{\tau}_1\big| + \alpha_2h_2(y) + \gamma_2 + m_1\big|\widetilde{\tau}_1\big| + \alpha_1h_1(x) + \gamma_1\Big) - k_0e_n^2 + e_n\overline{u}(t) \\ + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{20}$$

where $\alpha_1,\ \alpha_2,\ \gamma_1,\ \gamma_2$ are generally unknown positive values and $h_2(\cdot),\ h_1(\cdot)$ are known functions. If $\overline{u}(t)$ is defined as:

$$\overline{u}(t) = -sign(e_n)(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\alpha}_1h_1(x) + \hat{\alpha}_2h_2(y)). \tag{21}$$

Then (20) is converted to the following form:

$$\begin{aligned}\dot{V} \le |e_n|\Big(m_2\big|\widetilde{\tau}_2\big| + \widetilde{\alpha}_2h_2(y) + \widetilde{\gamma}_2 + m_1\big|\widetilde{\tau}_1\big| + \widetilde{\alpha}_1h_1(x) + \widetilde{\gamma}_1\Big) - k_0{e_n}^2 \\ + \sum_{i=1}^{2}(\widetilde{\alpha}_i\dot{\widetilde{\alpha}}_i + \widetilde{\gamma}_i\dot{\widetilde{\gamma}}_i + m_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i).\end{aligned} \tag{22}$$

If the adaptive rules are defined in the following form:

$$\dot{\widetilde{\alpha}}_1 = -|e_n|h_1(x) - \eta_1\widetilde{\alpha}_1,\ \ \dot{\widetilde{\alpha}}_2 = -|e_n|h_2(y) - \eta_2\widetilde{\alpha}_2,$$
$$\dot{\widetilde{\gamma}}_1 = -|e_n| - \eta_3\widetilde{\gamma}_1, \dot{\widetilde{\gamma}}_2 = -|e_n| - \eta_4\widetilde{\gamma}_2,$$
$$\dot{\widetilde{\tau}}_1 = -|e_n|sign(\widetilde{\tau}_1) - \eta_5\widetilde{\tau}_1, \dot{\widetilde{\tau}}_2 = -|e_n|sign(\widetilde{\tau}_2) - \eta_6\widetilde{\tau}_2.$$

Then, (22) becomes the following form:

$$\dot{V} \le -k_0 {e_n}^2 - \eta_1{\widetilde{\alpha}_1}^2 - \eta_2{\widetilde{\alpha}_2}^2 - \eta_3{\widetilde{\gamma}_1}^2 - \eta_4{\widetilde{\gamma}_2}^2 - \eta_5{\widetilde{\tau}_1}^2 - \eta_6{\widetilde{\tau}_2}^2 \le -\eta V. \tag{23}$$

Thus, $\eta = min(\eta_1,\ \eta_2, \ldots,\ \eta_6,\ k_0)$. Therefore, based on (23), it was verified that using the designed $u_0(t) = K^T E$ and $u(t)$ in form (11) and the adaptive rules determined in form (12), all the dynamics of synchronization error in differential equations are converged to zero asymptotically. In other words, condition (3) is met in Definition 1 and all estimation errors of the uncertain and unknown parameters $\widetilde{\alpha}_1,\ \widetilde{\alpha}_2,\ \widetilde{\gamma}_1,\ \widetilde{\gamma}_2,\ \widetilde{\tau}_1,\ \widetilde{\tau}_2$ have been converged to zero. □

4. Simulation Example and Results

In this section, applying the adaptive control strategy proposed in the previous section, the robust synchronization of two Jerk chaotic systems with uncertainty and bounded external disturbances as well as unknown time delays is simulated and evaluated. The state space equations of the Jerk system are described as follows [30]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= -\varepsilon_1 x_1(t) - x_2(t) - \varepsilon_2 x_3(t) - f(x_1(t)), \end{aligned} \tag{24}$$

where, ε_1 and ε_2 are system parameters and $f(.)$ is a piecewise linear function that is defined as follows:

$$f(x_1(t), t) = \frac{1}{2}(v_0 - v_1)\Big[\big|x_1(t) + 1\big| - \big|x_1(t) - 1\big|\Big] + v_1 x_1(t).$$

where, $v_0,\ v_1$ are constant values. Accordingly, the dynamic equations of Jerk chaotic master and slave systems based on Equation (24) in the presence of external disturbances, non-linear uncertainties and unknown time delays are presented in the following form:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= -\varepsilon_1 x_1(t) - x_2(t) - \varepsilon_2 x_3(t) - f(x_1(t-\tau_1), t) + \Delta f(x(t), t) + d_1(t), \end{aligned} \tag{25}$$

and the slave system is provided with a robust adaptive control strategy in the following form:

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= y_3(t) + u_0(t), \\ \dot{y}_3(t) &= -\varepsilon_1 y_1(t) - y_2(t) - \varepsilon_2 y_3(t) - g(y_1(t-\tau_2), t) + \Delta g(y(t), t) + d_2(t) + u(t). \end{aligned} \tag{26}$$

In the simulation process, the disturbances and boundary uncertainties of the system that satisfy the conditions of Assumption 1 are regarded as:

$$d_1(t) = (\sin(3t) + (\cos(2t))^2), \ \ d_2(t) = (0.2sin(t) + \sin(\pi t)), \tag{27}$$

$$\Delta f(x(t),t) = 0.1sin(x_1(t) + x_2(t) - 5x_3(t)), \ \Delta g(y(t),t) = 0.3sin(y_1(t) + y_2(t) - y_3(t)). \tag{28}$$

Additionally, piecewise linear functions with unknown time delays in the master and slave systems are illustrated as follows:

$$f(x_1(t-\tau_1),t) = \frac{1}{2}(v_0 - v_1)\Big[\big|x_1(t-\tau_1)+1\big| - \big|x_1(t-\tau_1)-1\big|\Big] + v_1 x_1(t-\tau_1), \tag{29}$$

$$g(y_1(t-\tau_2),t) = \frac{1}{2}(v_0 - v_1)\Big[\big|y_1(t-\tau_2)+1\big| - \big|y_1(t-\tau_2)-1\big|\Big] + v_1 y_1(t-\tau_2). \tag{30}$$

Established on Equations (25)–(30), the dynamic equations of synchronization error are defined as follows:

$$\begin{gathered} \dot{e}_1(t) = e_2(t), \\ \dot{e}_2(t) = e_3(t) + u_0(t), \\ \dot{e}_3(t) = -\varepsilon_1 e_1(t) - e_2(t) - \varepsilon_2 e_3(t) - g(y_1(t-\tau_2),t) + f(x_1(t-\tau_1),t) \\ +\Delta g(y(t),t) - \Delta f(x(t),t) + d_2(t) - d_1(t) + u(t). \end{gathered} \tag{31}$$

By describing $u_0(t) = k_1 e_1(t) + k_2 e_2(t) - e_3(t)$, the LTI segment of the synchronization error system (31) is defined as follows:

$$\begin{gathered} \dot{e}_1(t) = e_2(t), \\ \dot{e}_2(t) = k_1 e_1(t) + k_2 e_2(t). \end{gathered} \tag{32}$$

Therefore $k_1, \ k_2$ should be determined in such a way that according to Theorem 1 the error dynamics $e_1(t), \ e_2(t)$ in system (32) are stable and $u(t)$ based on a robust adaptive controller designed based on the adaptive rules in Theorem 2 are considered in form (11).

Accordingly, in order to evaluate the performance of the adaptive control strategy proposed for robust synchronization of systems (25) and (26), the initial conditions of the master and slave systems are equal to $[-0.5, \ 0.5, \ 0.87]$ and $[2, \ -1, \ -2]$, respectively, and are considered, the controller parameters $u_0(t)$ are equal to $k_1 = -1, \ k_2 = -2$ and in the controller (11), $k_0 = +1$. Moreover, to improve the controller's performance in detracting the output chattering rate, instead of the $sign(\cdot)$ function, the $tanh(\cdot)$ function is exploited in the controller structure (11). In accordance with the aforementioned, different simulation results are depicted in Figures 1–9.

Figures 1 and 2 exhibit the dynamic behavior of the master and slave systems in the absence of the proposed controllers indicating the dynamic behavior is chaotic due to the parameters defined above and external disturbances and uncertainties (27) and (28) applied to the systems. External disturbances and uncertainties defined in forms (27) and (28) applied into the master and slave systems with time delays of (25) and (26) are demonstrated in Figure 3.

In Figure 4, it is explicit that the synchronization of uncertain Jerk chaotic systems with unknown time delays (25) and (26) is adeptly carried out using the proposed adaptive control strategy, and this synchronization against system disturbances and uncertainties has profited from the desired robustness. As manifested in Figure 5, all synchronization errors have been converged to zero using the proposed robust and adaptive control strategy confirming the robust synchronization process of the Jerk master and slave systems.

The control signal based on the proposed adaptive controller for uncertainties and distortions applied on systems with unknown delay (25) and (26) is depicted in Figure 6. As can be perceived from Figure 6, the existent chattering rate in the proposed control signal is low portending the effective performance of the proposed controller in synchronizing the proposed chaotic uncertain systems. Finally, the estimation errors of the uncertain parameters as well as the unknown time delays in the master and slave systems are shown in Figure 6, which exhibits the achievement of the proposed control mechanism in estimating the parameters and updating them using the proposed adaptation rules with Equation (12).

Also, to evaluate the performance of the proposed control strategy, simulation of the synchronization process for master and slave systems (25), (26) with unknown variable time delays is shown in Figures 8 and 9. The results indicate the capability of the proposed control strategy for the aforementioned systems with unknown variable time delays.

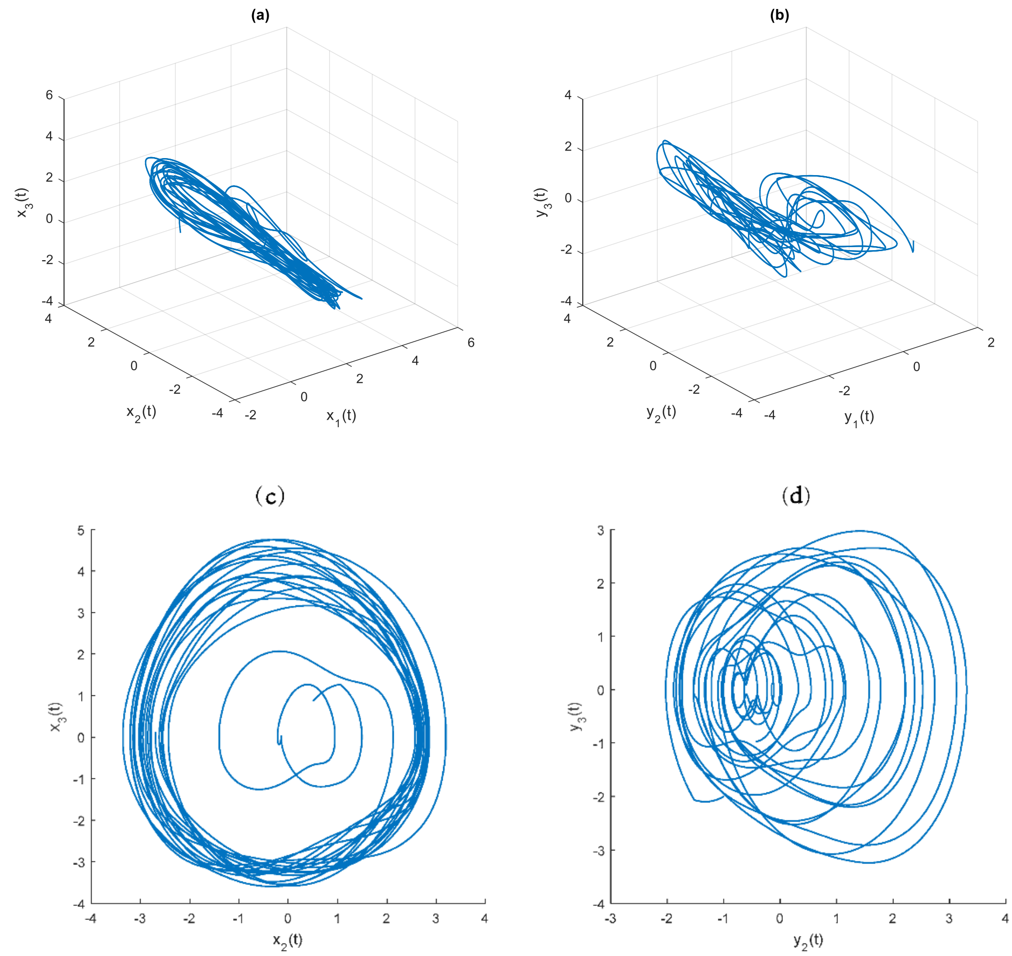

Figure 1. Exhibits chaotic behaviors of the fractional-order modified Jerk system without the controller: (a) The phase portrait of the master system (25); (b) the phase portrait of the slave system (26); (c) the plane x_2–x_3 of master system; and (d) the plane y_2–y_3 of the slave system.

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The control signal based on the proposed adaptive controller for uncertainties and distortions

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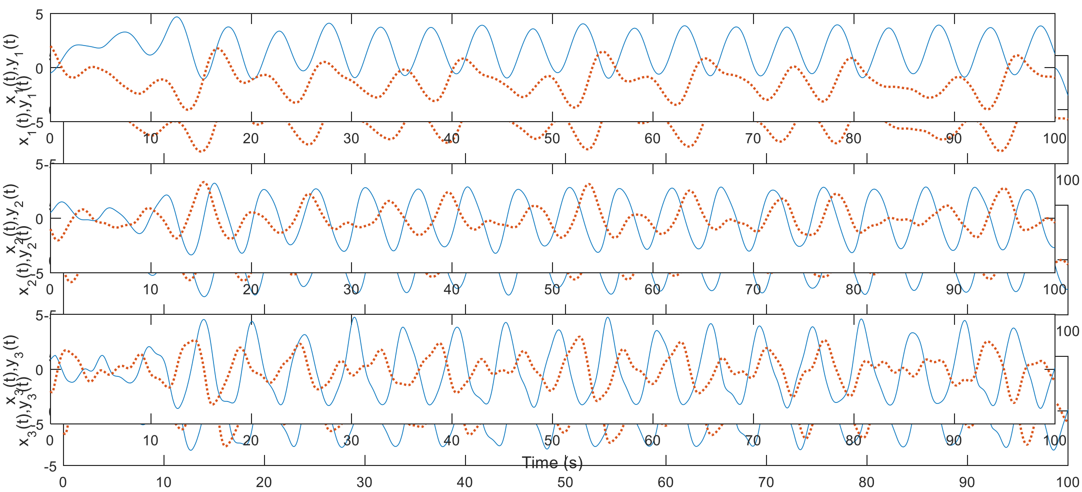

Figure 2. Representation of the dynamics of the master and slave systems (25), (26) without the controller. Solid lines are the dynamics of the master system and dotted lines are the dynamics of the slave system.

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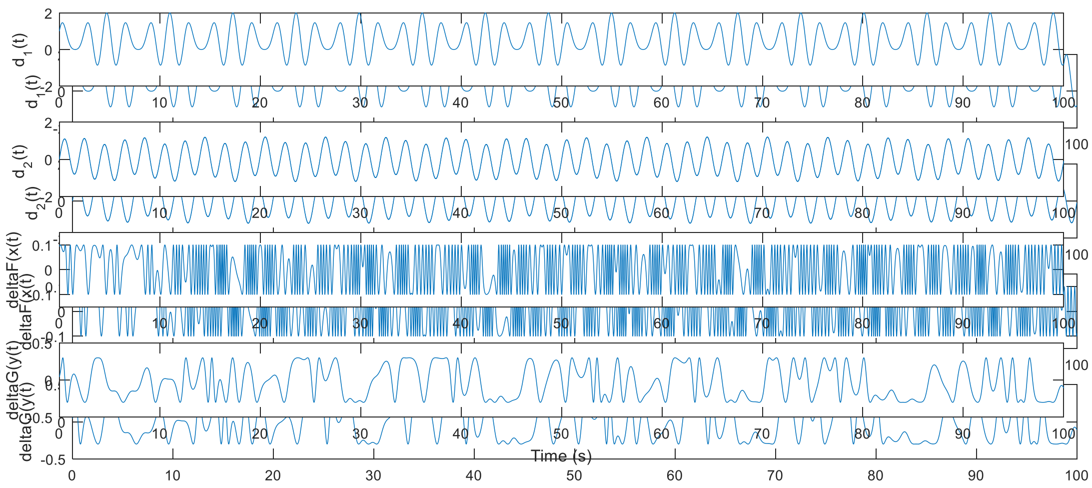

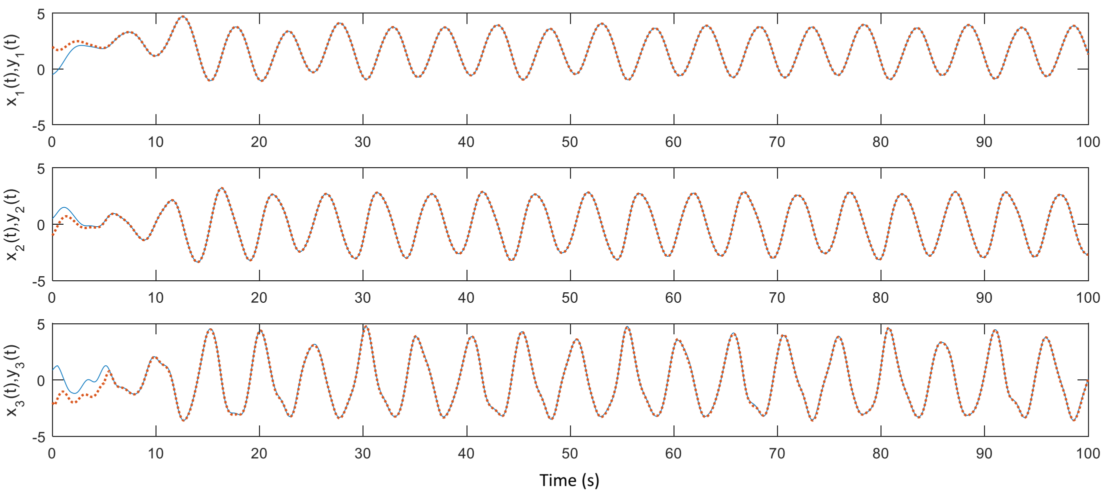

Figure 4. Representation of the robust synchronization presses of the master and slave systems (25), (26) based on the proposed adaptive control strategy where it is activated at $t = 5$ s.

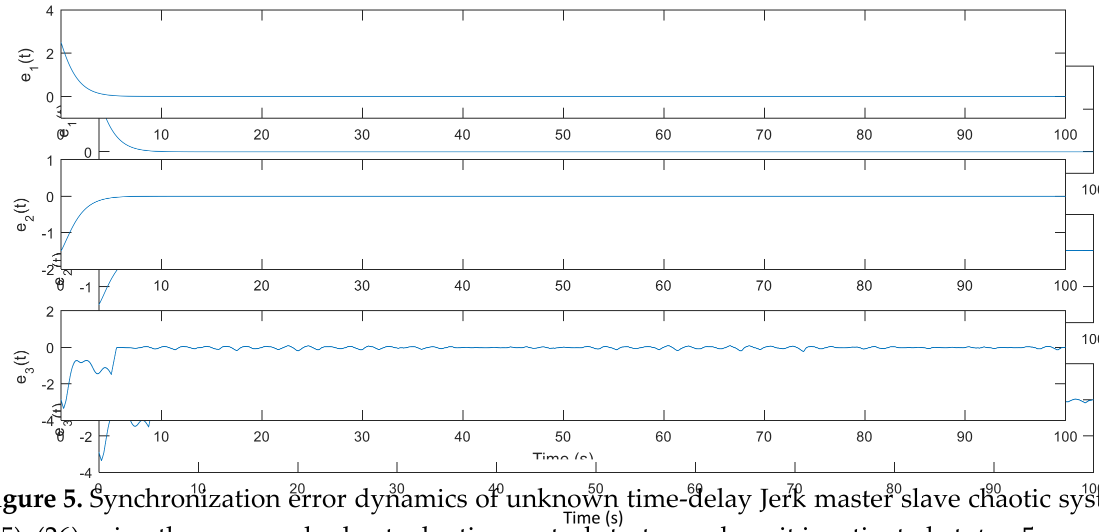

Figure 5. Synchronization error dynamics of unknown time-delay Jerk master slave chaotic systems (25), (26) using the proposed robust adaptive control strategy where it is activated at $t = 5$ s.

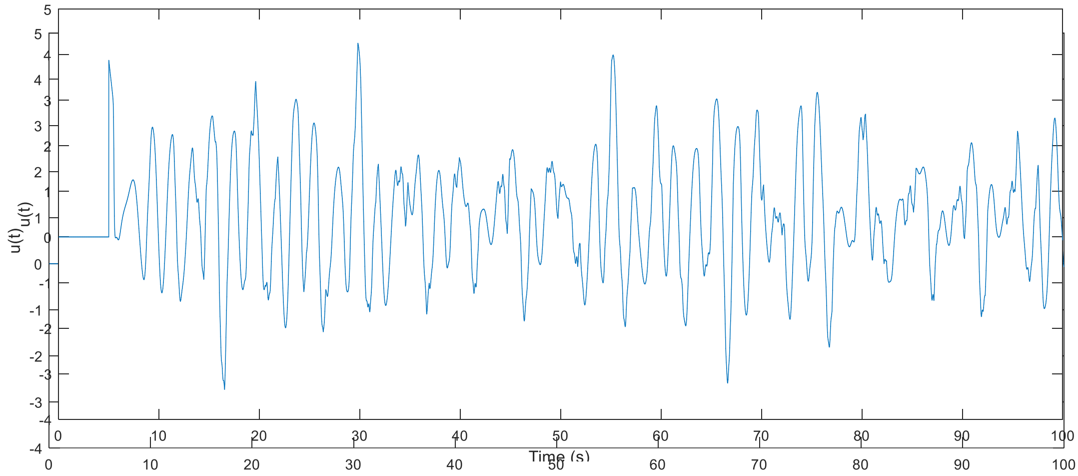

Figure 6. Control signal based on the proposed robust adaptive control strategy (11) where it is activated at $t = 5$ s.

Figure 6. Control signal based on the proposed robust adaptive control strategy (11) where it is activated at $t = 5$ s.

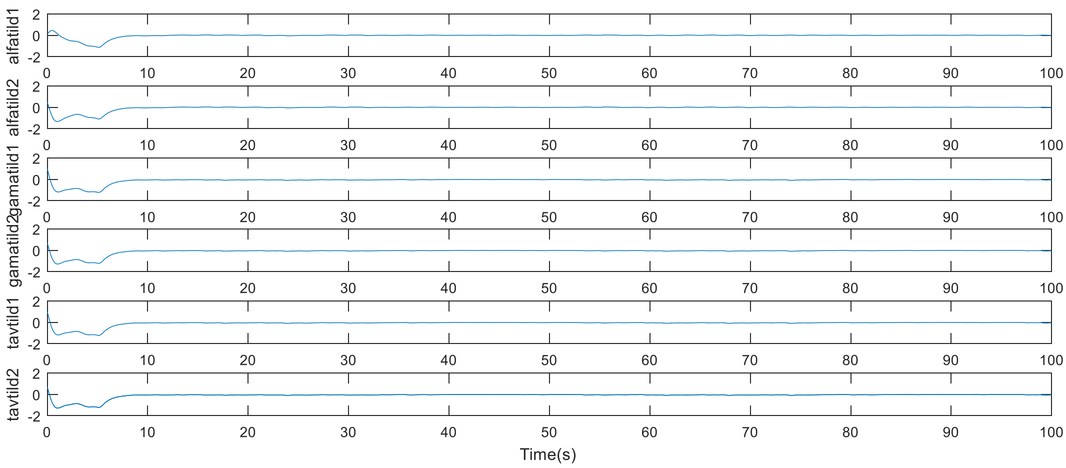

Figure 7. Estimation errors of unknown parameters in systems (25), (26) based on the proposed robust adaptive control strategy (11) where it is activated at $t = 5$ s, and the defined adaptation laws (12).

Figure 8. Estimation errors of unknown parameters in systems (25), (26) based on the proposed robust adaptive control strategy (11) where it is activated at $t = 5$ s, and the defined adaptation laws (12).

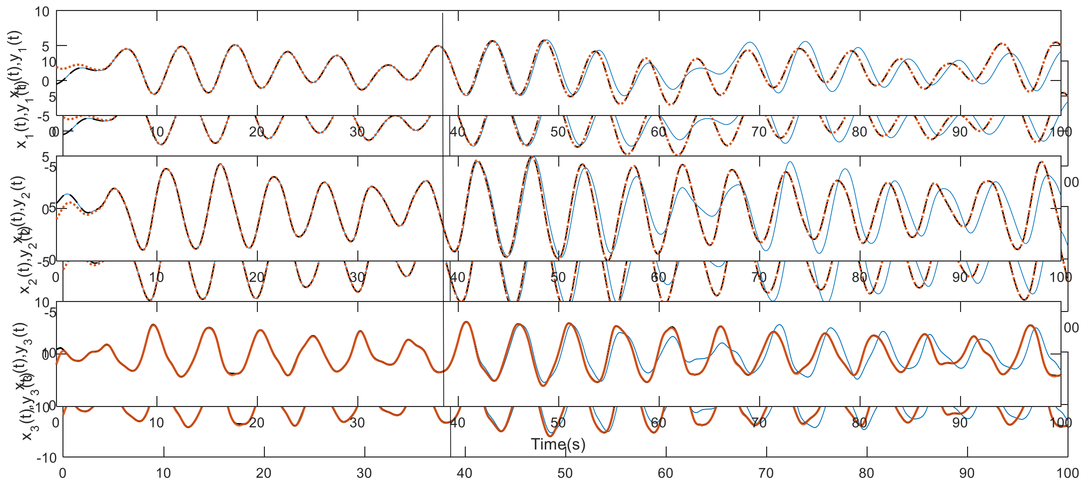

Figure 8. The robust synchronization presses of systems (25), (26) with unknown variable time delays based on the proposed adaptive control strategy where it is activated at $t = 5$ s. Solid lines are the dynamics of the master system for $\tau_1 = 0.45$, dashed-dotted lines are the dynamics of the master system based on step-change $\tau_1 = 0.45$ to $\tau_1 = 0.5$ at $t = 38.45$ s, and dotted lines are the dynamics of the slave system based on step-change $\tau_2 = 0.1$ to $\tau_2 = 0.7$ at $t = 38.45$ s.

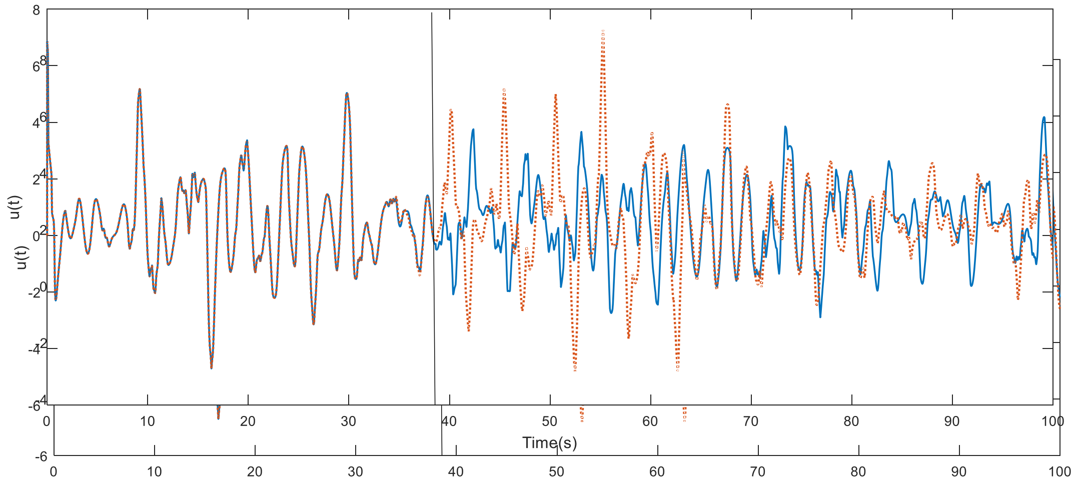

Figure 9. Controller signals based on the proposed robust adaptive control strategy (11) where it is activated at $t = 5$ s. The solid line is the control signal based on the fixed unknown time delays $\tau_1 = 0.45$, $\tau_2 = 0.1$, and the dotted line is the control signal based on step-change of unknown variable time delays $\tau_1 = 0.45$ to $\tau_1 = 0.5$ at $t = 38.45$ s and $\tau_2 = 0.1$ to $\tau_2 = 0.7$ at $t = 38.45$ s.

5. Conclusions

In this paper, a novel adaptive control strategy for robust synchronization of two uncertain chaotic systems with unknown time delays is presented. System uncertainties have been applied in the form of external disturbances and bounded non-linear uncertainties in master and slave chaotic systems with unknown time delays. In the proposed control mechanism, two controllers are utilized, and the robust stability of the controllers is proved by Lyapunov theory in the form of theorems. Adaptation rules for estimation of the uncertain parameters and unknown time delays in master and slave chaotic systems have also been proposed, and the capability of these rules to estimate uncertain parameters and unknown time delays has been proved using Lyapunov theory. Finally, to evaluate the proposed adaptive control mechanism, two uncertain jerk chaotic systems with unknown fixed and variable time delays have been simulated. The simulation results reveal the proficiency of the proposed method in robust synchronization of the aforementioned systems and desired estimation of uncertain parameters and unknown fixed and variable time delays of the systems.

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