ENHANCING THE RELIABILITY OF SERIES-PARALLEL SYSTEMS WITH MULTIPLE REDUNDANCIES BY USING SYSTEM-RELIABILITY INEQUALITIES

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Abstract

The reverse engineering of a valid algebraic inequality often leads to a projection of a novel physical reality characterized by a distinct signature: the algebraic inequality itself. This paper uses reverse engineering of valid algebraic inequalities for generating new knowledge and substantially improving the reliability of common series-parallel systems.

Our study emphasizes that in the case of series-parallel systems with interchangeable redundant components, the asymmetric arrangement of components always leads to higher system reliability than a symmetric arrangement. This finding remains valid, irrespective of the particular reliabilities characterizing the components.

Next, the paper presents novel system reliability inequalities whose reverse engineering enabled significant enhancement of the reliability of series-parallel systems with asymmetric arrangements of redundant components, without knowledge of the individual component reliabilities.

Lastly, the paper presents a new technique for validating complex algebraic inequalities associated with series-parallel systems. This technique relies on permutation of variable values and the method of segmentation.

Keywords: reliability, interpretation of algebraic inequalities, system reliability, reverse engineering of algebraic inequalities, optimisation, series-parallel systems, interchangeable redundancies

1. Introduction

Algebraic inequalities have been employed extensively to represent error bounds in approximations and constraints. Many valuable and non-trivial algebraic inequalities, along with their properties, have been recorded in a number of works [1-11]. Comprehensive range of techniques for proving algebraic inequalities has been covered in [3,4,6,8,10,11].

In engineering, algebraic inequalities have found diverse applications, as demonstrated in [12-15]. While some publications on algebraic inequalities [12] claim to address engineering applications, they primarily focus on using inequalities to formulate design constraints, establish upper and lower bounds, or explore the behaviour of mathematical functions, rather than optimizing engineering systems or processes.

In addition, inequalities have also played a crucial role in reliability and risk research, where they have been employed to provide bounds for reliability functions in various settings [16-22].

Algebraic inequalities represent a powerful tool for managing deep unstructured uncertainty, an aspect often overlooked by traditional approaches that primarily focus on structured uncertainty. The unique advantage of algebraic inequalities compared to other approaches [23] for dealing with unstructured uncertainty is their independence from specific variable values and the absence of any additional assumptions.

When the two sides of a valid algebraic inequality are physically interpreted as the output parameters of two distinct design configurations, the inequality can help determine which design configuration is superior with regard to that specific output parameter.

The effectiveness of algebraic inequalities in addressing deep unstructured uncertainty and design optimisation has been demonstrated in recent works by the author [24-26]. For example, in [24] systems were compared without knowledge of the reliabilities of their components. In [25], a comprehensive discussion of a method for design optimization based on the inverse approach has been provided, routed in the principle of non-contradiction. In a more recent work [26], the inverse approach has been used for maximizing the reliability of a series-parallel system when the components' reliabilities can be ranked. Nevertheless, the approach presented in [26] is limited to series-parallel systems with a single redundancy.

This paper extends the research in [26] by proposing new algebraic inequalities that are applicable to series-parallel systems with multiple redundancies.

The reliability of such systems is enhanced by using the inverse approach which effectively employs reverse engineering on correct algebraic inequalities. The reverse engineering on correct algebraic inequalities often leads to a projection of a novel physical reality characterised by a distinct signature: the algebraic inequality itself.

Different physical systems or processes may have a single algebraic inequality as a signature. In this respect, these physical systems and processes form many-to-one mapping with the inequality and new properties/behaviour related to different physical systems can be inferred from the same abstract inequality. Such is for example, the inequality of the additive ratios considered in [25].

Furthermore, the proposed approach for enhancing system reliability is domain-independent and does not require any information about the reliabilities of the components or their ranking. To prove the proposed system reliability inequalities, a novel method based on segmentation and permutation of the variable values has been developed.

It is important to note that the non-trivial knowledge presented in this paper cannot be obtained intuitively. The proposed algebraic inequalities are crucial for extracting the non-trivial knowledge necessary to enhance system reliability.

2. Improving the reliability of series-parallel systems with multiple redundancies, in cases of unknown reliabilities of components and symmetric arrangement of the redundancies

Reverse engineering of algebraic inequalities can be used to improve the reliability of series-parallel systems in case of unknown reliabilities of the components and symmetric arrangement of the redundancies. Each main component in the systems has the same number of interchangeable redundant components.

Consider the correct abstract algebraic inequality:

$$(1-x^3)(1-y^3)(1-z^3) \le (1-x^2y)(1-y^2z)(1-z^2x) \quad (1)$$

which will be proved rigorously in Section 4.In inequality (1): $0 \le x \le 1, 0 \le y \le 1,\ 0 \le z \le 1$.

Let the variables, x,y and z in inequality (1) be physically interpreted as 'probabilities of failure' of components of type X, Y and Z. It can be shown that, in this case, the left-hand side of inequality (1) can be physically interpreted as the reliabilities of two alternative series-parallel systems.

Indeed, consider the physical system in Fig.1. It features three pipelines with three valves of types X,Y and Z, physically arranged in series. All valves are initially open. With respect to stopping the fluid in all three pipelines, the reliability networks corresponding to the physical arrangements a) and b) are given by Figure 2a and Figure 2b, correspondingly. These reliability networks represent the logical arrangement of the valves, which is different from their physical arrangement. Each section of three valves in parallel in Figure 2 corresponds to the three valves in series on each of the pipelines in Fig.1.

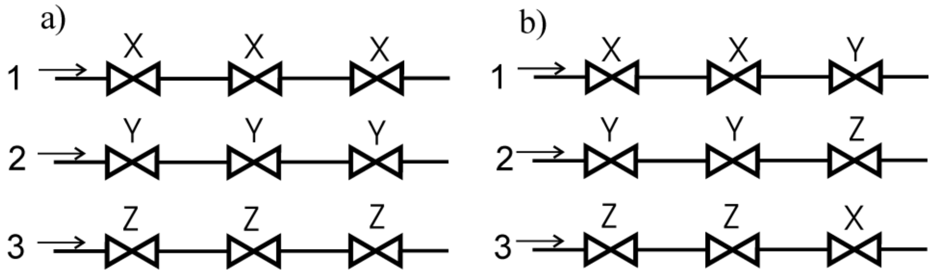

Figure 1. Functional diagrams of two different arrangements of valves on three pipelines.

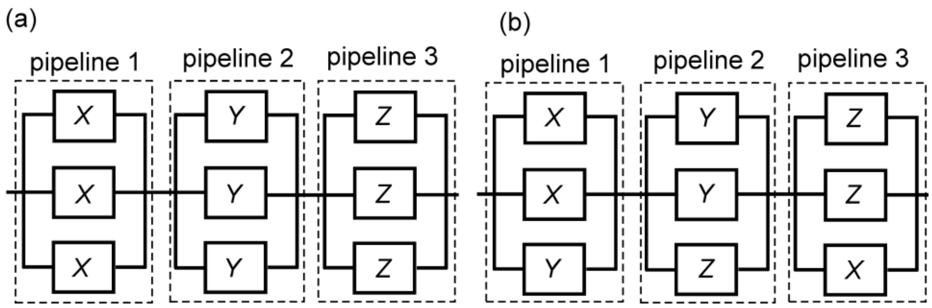

Figure 2. Reliability networks of the systems in Figure 1.

If the probabilities of failures of the valves of types X,Y and Z are denoted by *x*,*y* and *z*, respectively ($0 \le x \le 1, 0 \le y \le 1,\ 0 \le z \le 1$), the reliability R_a of the system in Figure 2a is given by $R_a = (1-x^3)(1-y^3)(1-z^3)$ while the reliability R_b of the system in Figure 2b is given by $R_b = (1-x^2 y)(1-y^2 z)(1-z^2 x)$.

The probabilities of failure *x*,*y* and *z*, of the valves are unknown.

The reverse engineering of inequality (1) yielded that the left-hand side of inequality (1) corresponds to the reliability of the system in Fig.2a while the right-hand side of the inequality corresponds to the reliability of the system in Figure 2b.

According to inequality (1), the reliability of the system in Figure 2b is superior to the reliability of the system in Figure 1a and this conclusion has been made *in total absence of knowledge regarding the probabilities of failure x,y and z of the three different types of valves*.

Inequalities similar to inequality (1) can be reverse engineered relatively easily because products of the type $(1-x_1^m)(1-x_2^m)...(1-x_n^m)$ where x_i is the probability of failure of component *i* can be interpreted directly as reliability of series-parallel systems with *m* components in parallel in each section in series.

In this connection, the inequalities (2)-(5), where $0 \le x, y, z \le 1$, can be proved rigorously and interpreted as reliabilities of series-parallel systems.

$$(1-x^2)(1-y^2)(1-z^2) \le (1-xy)(1-yz)(1-zx) \tag{2}$$

$$(1-x^3)(1-y^3)(1-z^3) \le (1-xyz)(1-yzx)(1-zxy) \tag{3}$$

$$(1-x^4)(1-y^4)(1-z^4) \le (1-x^2y^2)(1-y^2z^2)(1-z^2x^2) \tag{4}$$

$$(1-x^4)(1-y^4)(1-z^4) \le (1-x^2yz)(1-y^2zx)(1-z^2xy) \tag{5}$$

If *x*,*y* and *z* are interpreted as probabilities of failure of components from types *X*,*Y* and *Z*, the reverse engineering of inequalities (2)-(5) yields that their left- and right-hand sides correspond to the reliabilities of the series-parallel systems in Figures 3,4,5 and 6, correspondingly. The left hand-sides of the inequalities correspond to the systems 'a' while the right-hand sides correspond to systems 'b'.

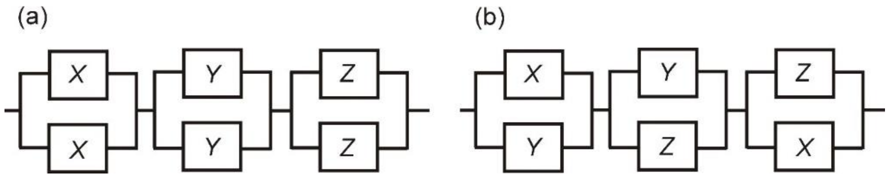

Figure 3. (a) A system obtained from the physical interpretation of the left-hand side of inequality (2); (b) A system obtained from the physical interpretation of the right-hand side of inequality (2).

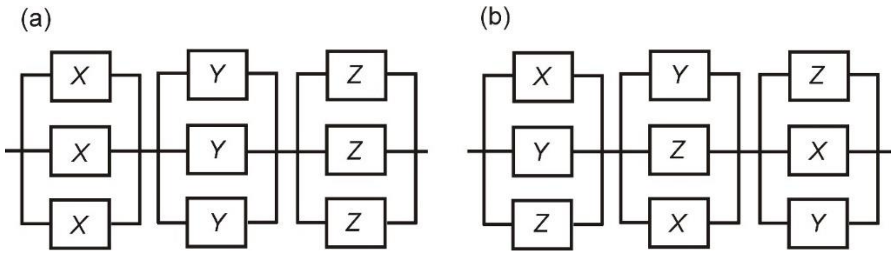

Figure 4. (a) A system obtained from the physical interpretation of the left-hand side of inequality (3); (b) A system obtained from the physical interpretation of the right-hand side of inequality (3).

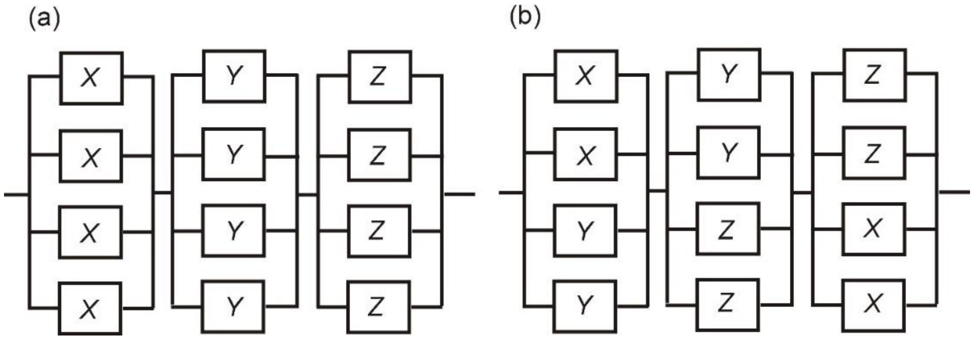

Figure 5. (a) A system obtained from the physical interpretation of the left-hand side of inequality (4); (b) A system obtained from the physical interpretation of the right-hand side of inequality (4).

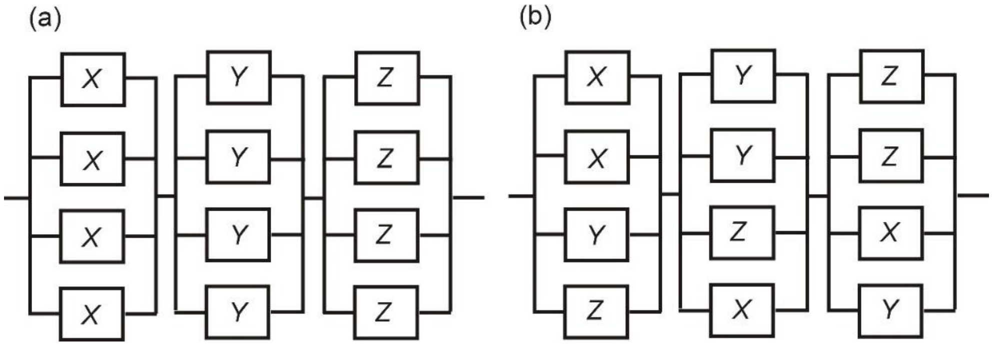

Figure 6. (a) A system obtained from the physical interpretation of inequality (5); (b) A system obtained from the physical interpretation of the right-hand side of inequality (5).

The reverse engineering of inequalities (1)-(5) yields the new knowledge that the system reliabilities of the asymmetric component arrangements 'b' is *always* greater than the system reliabilities of the symmetric component arrangements 'a'. As a result, inequalities (1)-(5) permit enhancing the reliability of series-parallel systems with multiple interchangeable redundant components in total absence of knowledge of the reliabilities of the components building the systems.

The system reliability enhancement can be dramatic, as the next numerical example demonstrates. Thus, for probabilities of failure $x = 0.08, y = 0.23, z = 0.91$ associated with a given period of operation, the reliability of the system configuration in Figure 2a is:

$$R_a = (1 - x^3)(1 - y^3)(1 - z^3) = (1 - 0.08^3)(1 - 0.23^3)(1 - 0.91^3) = 0.24$$

while the reliability of the system configuration in Figure 2b is

$$R_b = (1 - x^2 y)(1 - y^2 z)(1 - z^2 x) = (1 - 0.08^2 \times 0.23)(1 - 0.23^2 \times 0.91)(1 - 0.91^2 \times 0.08) = 0.89$$

The system reliability achieved through the asymmetric configuration of redundancies is 3.7 times greater than that of a symmetric arrangement!
These analyses indicate that in systems featuring interchangeable redundant components, a symmetric configuration of these components consistently results in lower system reliability compared to an asymmetric arrangement. This conclusion remains valid irrespective of the varying failure probabilities associated with different component types.

Additionally, the proposed approach is domain-independent and can be utilized in diverse systems that involve multiple redundant components. As such, the method, for example, can be implemented in systems that measure critical quantities such as temperature, pressure, concentration, fluid levels, and so on, across multiple zones, using parallel sensors in each zone.

The proposed approach, for instance, can be effectively utilized in multi-sensor systems, where several redundant sensors operate concurrently to collect data, guaranteeing uninterrupted input in the event of a sensor failure.

Consider a scenario where the temperature, pressure, concentration, or fluid level within each of two distinct zones, *A* and *B* (as illustrated in Figure 7a and 7b), must be measured and relayed to a control device (*C*). The control device then compares the readings obtained from both zones. If the disparity between the measurements in zones *A* and *B* surpasses a critical threshold, the control device activates a safety system to mitigate potential risks.

Within each zone, the sensors are logically arranged in parallel. To enable a valid comparison, signals from at least one sensor in both zone *A* and zone *B* must be received by the control device *C*.

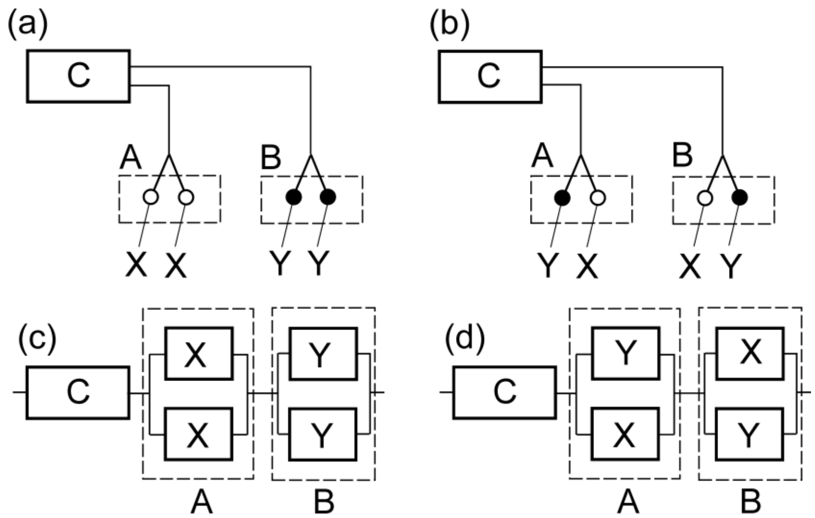

Figure 7. (a,b) Application example involving redundant sensors of type *X* and type *Y*; (c,d) Reliability networks corresponding to the systems in Fig.7a and 7b.

Suppose that the sensors are of two types: X and Y. The reliability networks of the systems in Figure 7a and 7b are provided in Figures 7c and 7d, respectively.

If $0 \le c \le 1$, $0 \le x \le 1$ and $0 \le y \le 1$ stand for the probabilities of failure of components C, X and Y, the reliability of the system configuration in Figure 7a is given by $R_a = (1-c)(1-x^2)(1-y^2)$ while the reliability of the system configuration in Figure 7b is given by $R_b = (1-c)(1-xy)(1-xy)$. The reverse engineering of the correct algebraic inequality

$$R_a = (1-c)(1-x^2)(1-y^2) \le R_b = (1-c)(1-xy)(1-xy)$$

yields that system reliability in Figure 7b consistently surpasses that of the system in Figure 7a. The last inequality holds true regardless of the individual probabilities of failure *x* and *y* of sensors type *X* and type *Y*, as well as the probability of failure *c* of the control device.

The difference in the system reliabilities of the alternatives in Figure 7a and 7b can be significant and this can be demonstrated with the specific probabilities of failure: $c = 0.1$, $x = 0.17$ and $y = 0.86$. The reliabilities of the alternative systems in Figure 7a and 7b are:

$R_a = (1-c)(1-x^2)(1-y^2) = (1-0.1)(1-0.17^2)(1-0.86^2) = 0.23$

and

$R_b = (1-c)(1-xy)(1-xy) = (1-0.1)(1-0.17 \times 0.86)^2 = 0.77$

The level of system reliability achieved through an asymmetric configuration of the sensors is 3.35 times greater than that of a symmetric arrangement.

In conclusion, it is not beneficial with respect to system reliability to place sensors of the same type within the same zone. Instead, mixing sensor types is recommended to achieve an asymmetrical configuration and ultimately enhance the overall system reliability.

The next application example comprises four parallel pipelines that transport toxic fluid. Each pipeline incorporates flanges sealed by two O-ring seals, with the second O-ring serving as a redundant seal. It's crucial to note that a solitary functional seal in a flange is sufficient to isolate the toxic fluid from the surrounding environment.

The O-ring seals utilized in this system are of two different materials: material *X* (represented by open circles) and material *Y* (represented by filled circles). For the system of four pipelines with flanges to operate reliably, it must ensure the absence of a leak of toxic fluid from any of the flanges.

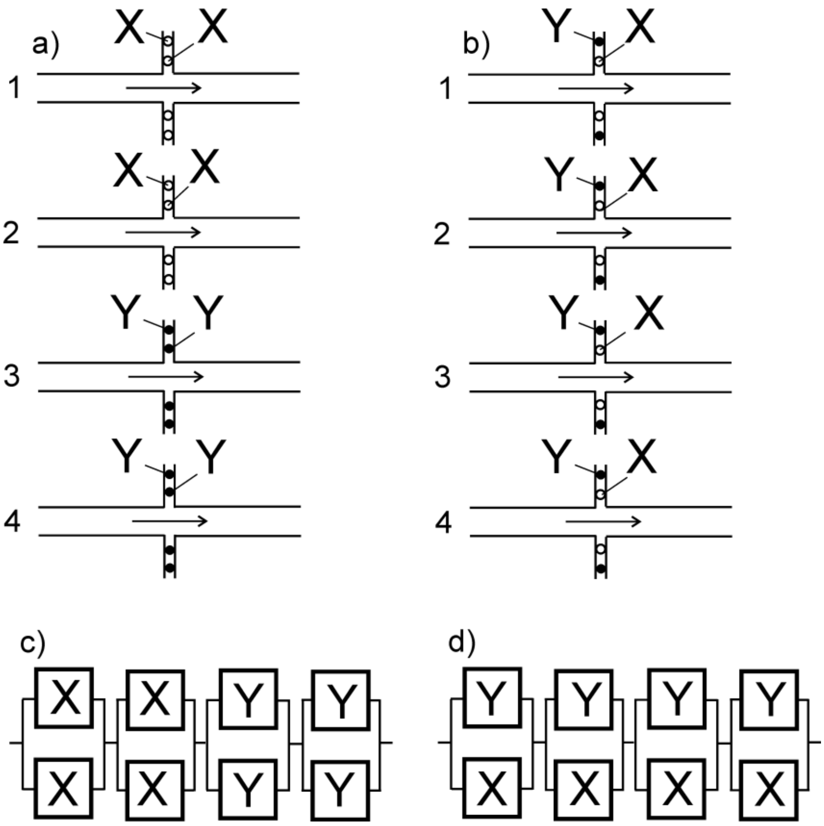

Figure 8. Application example involving four pipelines with flanges sealed with seals from two different materials *X* and *Y*; a) the original system with no permutation of the seals; (b) The system with superior reliability involving permutation of the seals; (c,d) Reliability networks corresponding to the systems in Fig.8a and 8b.

The flanges are logically arranged in series because all of the flanges must isolate the toxic fluid in order to prevent a release of toxic fluid in the environment. On the other hand, the seals within each flange are arranged in parallel because the reliable operation of just one seal is adequate to prevent a leak of the toxic fluid. As a result, the corresponding reliability networks for the systems illustrated in Fig.8a and Fig.8b can be found in Fig.8c and Fig.8d, respectively.

Let's denote the probability of failure of seals made from material *X* and *Y* as '*x*' and '*y*' respectively. The reliability of the seal system depicted in Fig.8a is then given by:

$R_a = (1-x^2)(1-x^2)(1-y^2)(1-y^2)$ and $R_b = (1-xy)(1-xy)(1-xy)(1-xy)$

The reverse engineering of the correct algebraic inequality:

$$R_a = (1-x^2)(1-x^2)(1-y^2)(1-y^2) \le R_b = R_b = (1-xy)(1-xy)(1-xy)(1-xy)$$

yields that the seals arrangement in Fig.8b is the more reliable seals arrangement, irrespective of the actual reliabilities of the seals from material *X* and *Y*. This algebraic inequality can be proved by using the simpler inequality $(1-x^2)(1-y^2) \le (1-xy)(1-xy)$ which is proved easily by expanding the brackets.

The conclusion is that mixing the seals from both types in each flange yields a more reliable system.

The system reliability enhancement can be dramatic, as the next numerical example demonstrates. Thus, for probabilities of failure $x = 0.17, y = 0.86$ related to the seals of type *X* and type *Y*, correspondingly, the reliability of the system configuration in Figure 8a is:

$$R_a = (1-x^2)^2(1-y^2)^2 = (1-0.17^2)(1-0.86^2)^2 = 0.064$$

while the reliability of the system configuration in Figure 8b is

$$R_b = (1-xy)^2(1-yx)^2 = (1-0.17\times 0.86)^2(1-0.86\times 0.17)^2 = 0.53$$

The system reliability achieved through the asymmetric configuration of redundancies is 8.3 times greater than that of a symmetric arrangement!

3. Improving the reliability of series-parallel systems with multiple redundancies in case of asymmetric arrangement of the redundancies

Consider now the algebraic inequality:

$$(1-x^2y)(1-y^2z)(1-z^2x) \le (1-xyz)(1-xyz)(1-xyz) \qquad (6)$$

which will be proved rigorously in Section 4.

The reverse engineering of this inequality yields a physical interpretation of the left-hand side of inequality (6) as the reliability of the system in Fig.9a while the physical interpretation of the right-hand side yields the reliability of the system in Figure 9b.

The physical system in Fig.9a features three pipelines with three valves of types X,Y and Z, physically arranged in series. All valves are initially open. With respect to stopping the fluid in all three pipelines, the reliability networks corresponding to the asymmetric physical arrangements a) and b) are given by Figure 10a and Figure 10b, correspondingly. Each section of three valves in parallel in Figure 10 corresponds to the three valves in series on each of the pipelines in Figure 9. If the probabilities of types X,Y and Z valves are denoted by *x*,*y* and *z*, respectively, the reliability of the system in Figure 10a is given by $R_a = (1-x^2y)(1-y^2z)(1-z^2x)$ while the reliability of the system in Figure 10b is given by $R_b = (1-xyz)(1-xyz)(1-xyz)$.

According to the physical interpretation of inequality (6), the reliability of the system in Figure 10b is superior to the reliability of the system in Figure 10a and this conclusion has been made in total absence of knowledge regarding the probabilities of failure *x*,*y* and *z* of the valves.

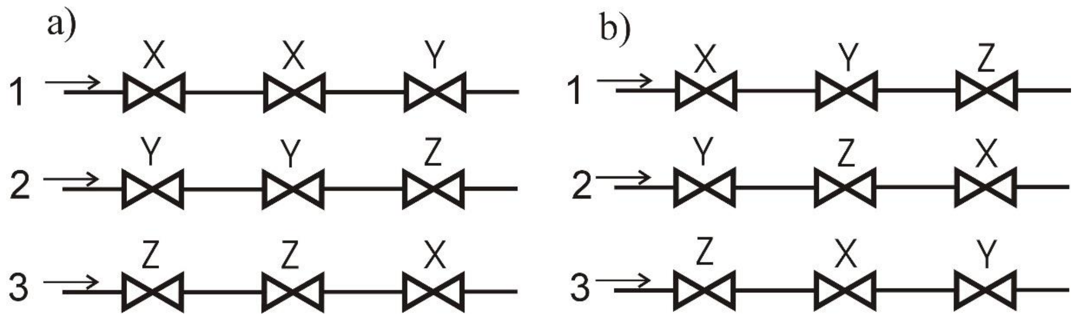

Figure 9. Functional diagrams of two different arrangements of valves on three pipelines. Each pipeline includes valves of different type.

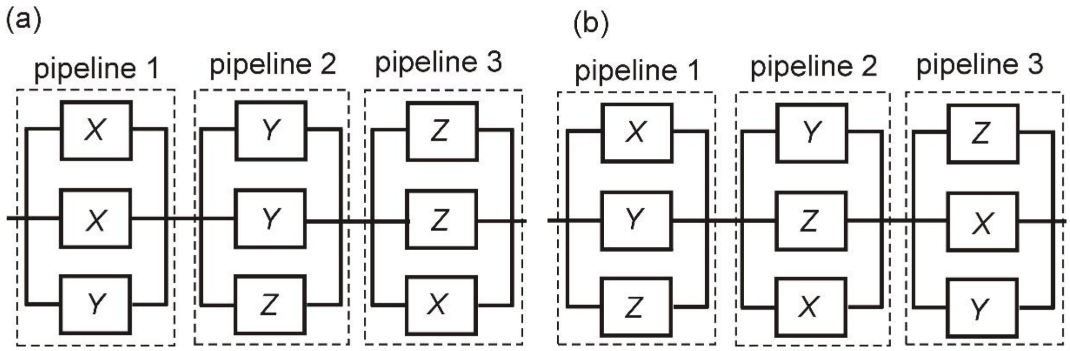

Figure 10. Reliability networks of the systems in Figure 9.

The following inequalities can be proved rigorously and their sides can also be physically interpreted as reliabilities of physical systems:

$$(1-x^3y)(1-y^3z)(1-z^3x) \le (1-x^2yz)(1-y^2zx)(1-z^2xy) \tag{7}$$

$$(1-x^2y^2)(1-y^2z^2)(1-z^2x^2) \le (1-x^2yz)(1-y^2zx)(1-z^2xy) \tag{8}$$

$$(1-x^4y)(1-y^4z)(1-z^4u)(1-u^4x) \le (1-x^2yzu)(1-xy^2zu)(1-xyz^2u)(1-xyzu^2) \tag{9}$$

where $0 \le x, y, z, u \le 1$.

If *x,y,z,u* are interpreted as probabilities of failure of components X,Y,Z,U, the left and right-hand sides of inequalities (6)-(9) correspond to the reliabilities of the series-parallel systems in Figures 10,11,12 and 13, correspondingly.

The left hand-sides of the inequalities correspond to the reliabilities of the systems 'a' while the right-hand sides correspond to the reliabilities of systems 'b'.

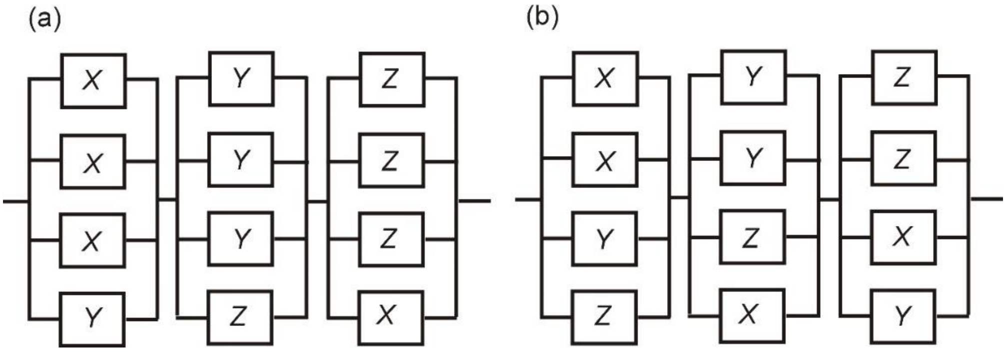

Figure 11. (a) A system obtained from the physical interpretation of the left-hand side of inequality (7); (b) A system obtained from the physical interpretation of the right-hand side of inequality (7).

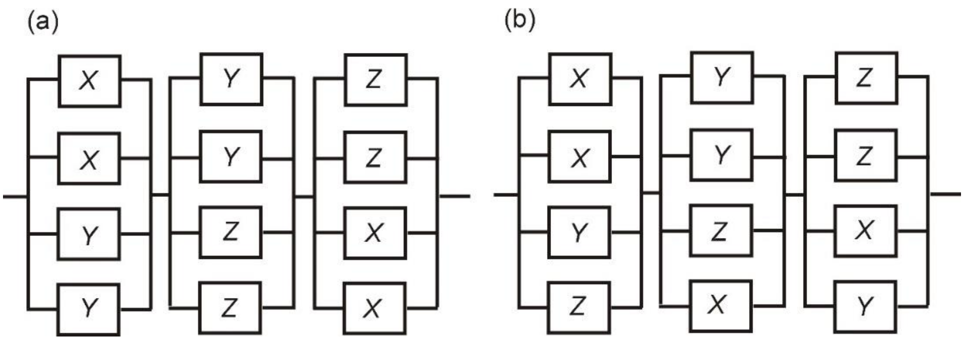

Figure 12. (a) A system obtained from the physical interpretation of the left-hand side of inequality (8); (b) A system obtained from the physical interpretation of the right-hand side of inequality (8).

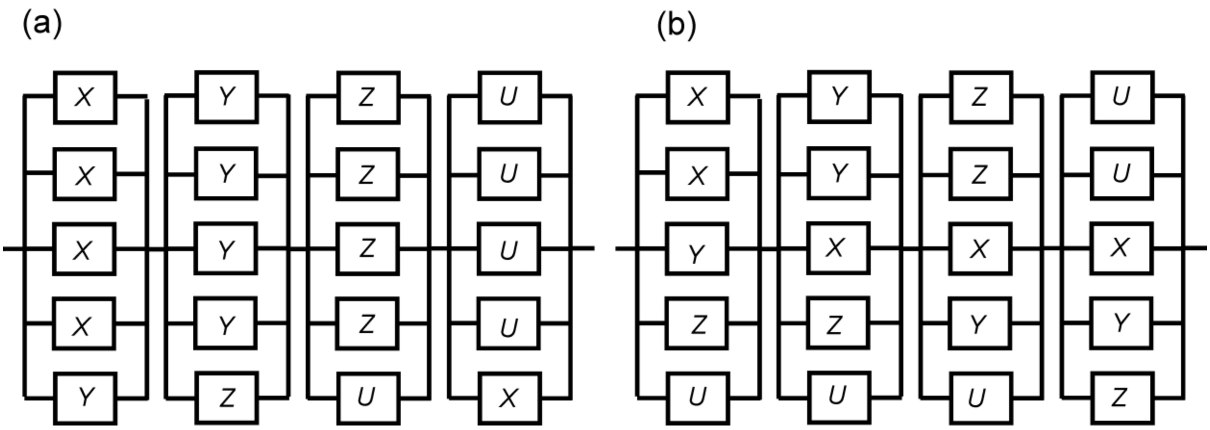

Figure 13. (a) A system obtained from the physical interpretation of the left-hand side of inequality (9); (b) A system obtained from the physical interpretation of the right-hand side of inequality (9).

The reverse engineering of the correct algebraic inequalities (6)-(9) yields the new knowledge that the system reliabilities of the asymmetric arrangements of redundant components in systems 'b' is *always* greater than the system reliabilities of the asymmetric arrangements of the components in systems 'a', *irrespective of the probabilities of failure of the separate components*. As a result, the reverse engineering of these inequalities permits enhancing the reliability of series-parallel systems with asymmetric arrangement of the components constituting the systems without requiring any knowledge of the individual reliabilities of the components or their ranking.

The system reliability enhancement can be dramatic, as the next numerical example demonstrates. Thus, for probabilities of failure $x = 0.17, y = 0.77, z = 0.86$ associated with a given period of operation, the reliability of the system configuration in Figure 10a is:

$$R_a = (1 - x^2 y)(1 - y^2 z)(1 - z^2 x) = (1 - 0.17^2 \times 0.77)(1 - 0.77^2 \times 0.86)(1 - 0.86^2 \times 0.17) = 0.42$$

while the reliability of the system configuration in Figure 10b is

$$R_b = (1 - xyz)(1 - xyz)(1 - xyz) = (1 - 0.17 \times 0.77 \times 0.86)^3 = 0.7$$

The reorganization of redundancies through an alternative arrangement has successfully improved system reliability by a factor of 1.67!

4. A technique for proving system reliability inequalities based on segmentation and variable values permutation

The system reliability inequalities (1)-(9) can be proved by a novel technique based on segmentation and variable values permutation. This technique will be illustrated by proving inequalities (1) and (6).

Proof of inequality (1).

The proof of the complex inequality (1) starts with proving the simpler inequalities

$$(1 - x^3)(1 - y^3) \le (1 - x^2 y)(1 - y^2 x) \tag{10}$$

$$(1 - y^3)(1 - z^3) \le (1 - y^2 z)(1 - z^2 y) \tag{11}$$

$$(1 - z^3)(1 - x^3) \le (1 - z^2 x)(1 - x^2 z) \tag{12}$$

These simpler inequalities can be viewed as building segments whose multiplication leads to an inequality closely related to inequality (1).

Inequalities (10)-(12) are easily proved by direct manipulation. For example, proving inequality (10) is equivalent to proving the inequality $x^3 + y^3 \ge x^2 y + y^2 x$, which is equivalent to proving the inequality $x^3 - x^2 y + y^3 - y^2 x \ge 0$ or $x^2(x - y) - y^2(x - y) \ge 0$. The last inequality is equivalent to $(x-y)^2(x+y)\geq 0$ which is always true considering that *x* and *y* are non-negative.

Because *x*,*y* and *z* are non-negative numbers between 0 and 1, the left- and right-hand sides of inequalities (10),(11) and (12) are all non-negative. Multiplying the left- and right-hand sides of inequalities (10),(11) and (12) therefore will not alter the direction of the resultant inequality. Consequently, multiplying the left and right-hand parts of inequalities (10)-(12) results in a correct inequality with the same direction:

$$[(1-x^3)(1-y^3)(1-z^2)]^2 \leq (1-x^2y)(1-y^2z)(1-z^2x)(1-y^2x)(1-z^2y)(1-x^2z) \tag{13}$$

that holds for any *x*,*y* and *z* for which $0\leq x,y,z\leq 1$.

However, inequality (13) can only be true if and only if both inequalities (14) and (15) are satisfied simultaneously.

$$(1-x^3)(1-y^3)(1-z^3)\leq(1-x^2y)(1-y^2z)(1-z^2x) \tag{14}$$

$$(1-x^3)(1-y^3)(1-z^3)\leq(1-y^2x)(1-z^2y)(1-x^2z) \tag{15}$$

Note that the direction of both inequalities (14) and (15) cannot be '$\geq$' because, in this case, the product of the left- and right-hand sides of inequalities (14) and (15) would generate an inequality with a direction opposite to that of inequality (13).

Suppose that only the direction of inequality (15) is reversed. Hence, according to our assumption, the inequality

$$(1-x^3)(1-y^3)(1-z^3)\geq(1-y^2x)(1-z^2y)(1-x^2z) \tag{16}$$

holds for any *x*,*y* and *z* for which $0\leq x,y,z\leq 1$.

We aim to demonstrate that this particular assumption leads to a contradiction. To do so, we give the specific values $x=a, y=b, z=c$ for the variables *x*,*y*,*z* and from inequality (14) we get:

$$(1-a^3)(1-b^3)(1-c^3)\leq(1-a^2b)(1-b^2c)(1-c^2a) \tag{17}$$

Now we interchange the values of variables *x* and *y*: $x=b, y=a, z=c$. From inequality (15) we get

$$(1-b^3)(1-a^3)(1-c^3)\geq(1-a^2b)(1-c^2a)(1-b^2c) \tag{18}$$

The left and right sides of inequalities (17) and (18) are identical; however, the direction of inequality (18) is opposite to that of inequality (17). This contradiction is a result of the assumption made about the direction of inequality (16). Consequently, the direction of inequality (16) is the same as the direction of inequality (15).

In order to conserve space, we will refrain from reiterating the analogous arguments employed in the proof of inequalities (2) through (5). Additionally, these inequalities can be naturally extended to more than three variables; however, the specifics of this generalization process have also been omitted for the sake of brevity.

Proof of inequality (6).

The poof of the complex inequality (6) starts with proving the simpler inequalities

$$(1-x^2y)(1-z^2y)\le(1-xyz)^2 \tag{19}$$

$$(1-y^2z)(1-x^2z)\le(1-xyz)^2 \tag{20}$$

$$(1-z^2x)(1-y^2x)\le(1-xyz)^2 \tag{21}$$

whose multiplication yields an inequality closely related to the original inequality (6).

The simpler inequalities (19)-(21) are proved by direct manipulation. For example, proving inequality (19) is equivalent to proving the inequality $z^2y+x^2y\ge xyz+xyz$, which is equivalent to proving the inequality $z^2y+x^2y-xyz-xyz\ge 0$ or $zy(z-x)-xy(z-x)\ge 0$. The last inequality is equivalent to $(z-x)^2y\ge 0$ which is always true.

Because x,y and x are positive numbers between 0 and 1, the left- and right-hand sides of inequalities (19),(20) and (21) are all non-negative.

Consequently, multiplying the left and right-hand sides of inequalities (19)-(21) results in a correct inequality with the same direction:

$$[(1-x^2y)(1-y^2z)(1-z^2x)]\times[(1-y^2x)(1-z^2y)(1-x^2z)]\le[(1-xyz)(1-xyz)(1-xyz)]^2 \tag{22}$$

However, inequality (22) can only be true if and only if both inequalities (23) and (24) are satisfied simultaneously.

$$(1-x^2y)(1-y^2z)(1-z^2x)\le(1-xyz)(1-xyz)(1-xyz) \tag{23}$$

$$(1-y^2x)(1-z^2y)(1-x^2z)\le(1-xyz)(1-xyz)(1-xyz) \tag{24}$$

Again, note that the direction of both inequalities (23) and (24) cannot be '$\ge$' because, in such a scenario, the product of the left- and right-hand sides of inequalities (23) and (24) would result in an inequality with a direction contrary to that of inequality (22).

Suppose that only the direction of inequality (24) is reversed. As a result, according to our assumption, the inequality

$$(1-y^2x)(1-z^2y)(1-x^2z)\ge(1-xyz)(1-xyz)(1-xyz) \tag{25}$$

holds.

We aim to demonstrate that this particular assumption leads to a contradiction. To do so, we give the specific values $x = a, y = b, z = c$ and from inequality (23) we get:

$$(1-a^2b)(1-b^2c)(1-c^2a) \leq (1-abc)(1-abc)(1-abc) \tag{26}$$

Now we interchange the values of variables *x* and *y*: $x = b, y = a, z = c$. From inequality (25) we get

$$(1-a^2b)(1-c^2a)(1-b^2c) \geq (1-abc)(1-abc)(1-abc) \tag{27}$$

The left- and right-hand sides of inequalities (26) and (27) are identical; however, the direction of inequality (27) is opposite to that of inequality (26). This contradiction is a result of the assumption made about the direction of inequality (24). Consequently, the direction of inequality (24) is the same as the direction of inequality (23).

To conserve space, we will not restate the identical arguments used to prove the remaining inequalities (7) through (9). Moreover, the inequalities can readily be extended to encompass more than three variables, with the details of the generalization process left out.

It is important to highlight that the system reliability ranking of the series-parallel systems depicted in Figures 11, 12, and 13 remains valid, regardless of the failure probabilities of the individual components building these systems.

It is important to note that not all abstract algebraic inequalities have a direct physical interpretation. Consider the general inequality

$$\varphi_1(x,y,z) + \varphi_2(x,y,z) + \varphi_3(x,y,z) > \psi(x,y,z) \tag{28}$$

correct for any positive *x*,*y* and *z*. In this inequality, $\varphi_i()$ and $\psi()$ stand for particular functions of the variables *x*,*y* and *z*.

Suppose that inequality (28) is interpreted as a physical system or process and the variables *x*,*y*, and *z* have a particular physical interpretation.

To make sense for a physical system or process all terms $\varphi_i(x,y,z)$ in the left-hand side must have physical meaning and the same units, otherwise the terms $\varphi_i(x,y,z)$ in the left-hand side simply cannot be added together. In addition, the quantity $\psi(x,y,z)$ in the right-hand side, must also have a physical meaning and the same units as the left-hand side. Alternatively, all additive terms of the inequality must be dimensionless and have physical meaning.

For a large number of correct abstract algebraic inequalities, these requirements are too restrictive and for this reason no physical meaning can be attached to such inequalities.

CONCLUSIONS

1. In series-parallel systems with multiple interchangeable redundant components, the system reliability achieved through an asymmetric arrangement of the redundant components is always superior to the system reliability achieved through a symmetric arrangement of the redundant components. This result holds regardless of the failure probabilities of the different types of components.
2. By reverse engineering of algebraic inequalities, insights have been gained that have led to marked enhancements in the reliability of series-parallel systems with interchangeable redundancies. This improvement has been achieved without any knowledge of the individual component reliabilities within the systems.
3. A new technique for proving complex algebraic inequalities related to the reliability of series-parallel systems with interchangeable redundancies has been developed and demonstrated. The technique relies on permutation of variable values and the method of segmentation.
4. The reverse engineering of a valid algebraic inequality often leads to a projection of a novel physical reality characterized by a distinct signature: the algebraic inequality itself.
5. The proposed approach is domain-independent and can be utilized in diverse systems that involve multiple interchangeable redundant components. For instance, the method is particularly well-suited for systems that include different types of valves or multi-sensor measurements with sensors of different kinds.

These results have important practical implications for engineering and system design, as they offer a new approach to greatly improving the reliability of many systems. By optimizing the arrangement of interchangeable redundant components, it is possible to dramatically enhance the system's overall performance and durability, thus reducing the likelihood of costly failures and downtime.

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