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A fracture condition incorporating the most unfavourable orientation of the crack

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ABSTRACT: A fracture condition incorporating the most unfavourable orientation of the crack has been derived to improve the safety of loaded brittle components with complex shape, whose loading results in a three- dimensional stress state. With a single calculation, an answer is provided to the important question whether a randomly oriented crack at a particular location in the stressed component will cause fracture.

Brittle fracture is a dangerous failure mode and requires a conservative design calculation. The presented experimental results show that during a mixed-mode fracture the locus of stress intensity factors which result in fracture is associated with significant uncertainty. Consequently, a new approach to design of safety-critical components has been proposed, based on a conservative safe zone, located away from the scatter band defining fracture states. A postprocessor based on the proposed fracture condition and conservative safe zone can be easily developed, for testing loaded safety-critical components with complex shape. For each finite element, only a single computation is made which guarantees a high computational speed. This makes the proposed approach particularly useful for incorporation in a design optimisation loop.

Keywords: brittle fracture, design criterion, mixed-mode, vulnerability

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1. Introduction

Unlike ductile fracture, brittle fracture occurs suddenly, proceeds at a high speed and in order to progress, there is no need for the loading stress to increase (Anderson, 2005; Ewalds, and Wanhill 1984; Hertzberg, 1996). Brittle fracture also requires a relatively small amount of accumulated strain energy. These features make brittle fracture a dangerous failure mode and require a conservative approach to the design of safety-critical brittle components.

Vulnerability to brittle failure initiated by flaws, is a common type of mechanical vulnerability. It is a susceptibility to brittle failure caused by a combination of a flaw with size just below the threshold detection limit of the non-destructive inspection technique, an unfavourable location of the flaw in a high-stress region and unfavourable orientation of the flaw with respect to the local stresses. However, the capability of existing design methods to detect vulnerability to brittle failure initiated by flaws is limited. The standard design approach is to place a sharp crack with size equal to the threshold detection limit of the non-destructive inspection technique, in the most dangerous position and in the most dangerous orientation (where the stress takes its maximum value) and to test by using a fracture criterion whether the sharp crack will be unstable. This approach however works only in cases of components with simple shape and loading characterised by a one-dimensional or two-dimensional stress state. In this case, it is relatively easy to identify the most dangerous orientation of the crack. In cases of components with complex shape, loaded in complex fashion, as a rule, the stress state is three-dimensional and it is near to impossible for engineer-designers to identify the most unfavourable orientation of the crack associated with the highest driving force for crack extension. The space of possible orientations of the crack with respect to a three-dimensional stress tensor is huge, which makes it practically impossible to identify the most unfavourable orientation, even after a substantial number of empirical trials. Resolving this predicament and supplying the designer with a simple test for a mixed-mode fracture, incorporating the most unfavourable orientation of the crack, is the main purpose of this paper.

To reveal the vulnerability to brittle fracture, it can be assumed that the component under consideration contains a sharp penny-shaped crack with size equal to the threshold flaw size of the non-destructive inspection technique. It is also assumed that the sharp crack can reside anywhere in the component. The reliability-critical parameters are the random location of the crack and its random orientation. The random location defines the local stress state, while the random orientation defines the actual normal and shear stresses acting on the crack plane at the sampled location. After sampling a random location and a random orientation, an empirical mixed-mode fracture criterion (Dowling, 1999; Lee and Advani 1982; Lim et al. 1994; Huang and Lin 1996), can be employed to determine whether there will be a brittle fracture.

This simple approach however has severe limitations. Very often, the volumes of the stress concentration zones, where the dangerous stress states can be found, are too small in comparison with the total volume of the specimen. Generating a random location in the loaded component does not necessarily guarantee that each stress concentration zones will be sampled a sufficient number of times or that they will be sampled even once. Another complication is created by the circumstance that even if the small stress concentration zones are sampled, the generated random crack orientation may be benign for the sampled location, in which case the used fracture criterion will fail to 'register' fracture. The usual way of obtaining the stress variation in loaded component is by using a finite element solver. In order to avoid missing a dangerous location and a dangerous combination of a crack location and crack orientation, each of the finite elements must be visited. Next, by using the principal stresses characterising the visited finite element, a check can be performed whether a penny-shaped crack with a threshold detection size will cause failure for each possible random orientation. Testing for brittle fracture at each possible random orientation of the penny-shaped crack, at a given location, is a time consuming task. Suppose that 1000 random orientations are tested at each random location. In this case, the analysis of a loaded component including hundreds of thousands of finite elements will involve hundreds of millions of calculations with the selected failure criterion. Such a computation will require a large amount of time even on a very fast computer. The answer to this predicament is a failure condition which incorporates the most unfavourable crack orientation, at any specified location. In this case, only a single calculation involving the fracture criterion will be made (instead of 1000 calculations), for each random location. As a result, the computational speed will be increased by many orders of magnitude.

2. Defining the safe zone in the design of brittle components.

Brittle fracture is a dangerous failure mode and requires selecting a conservative fracture criterion. A combination of a high-magnitude normal stress and a high-magnitude shear stress is often present for certain orientations of the threshold crack and both, the stress intensity factor K_I characterising the tensile crack opening mode and the stress intensity factor K_{II} characterising the sliding crack opening mode make a significant contribution towards the fracture initiation. A number of empirical mixed-mode criteria reflecting this joint contribution have already been proposed. They have the common form

$$(K_I / K_{Ic})^p + (K_{II} / K_{IIc})^q = 1 \quad (1)$$

where K_{Ic} and K_{IIc} are the average fracture toughness values characterising the tensile and sliding opening mode, respectively (Anderson, 2005; Ewalds and Wanhill 1984). The constants $p \geq 1$ and $q \geq 1$ depend only on the material properties.

Various values for the material constants *p* and *q* are examined and the ones providing the best fit to the experimental results are selected (Lim et al., 1994; Dowling, 1999; Chang et al., 1995; Broek, 1986; Paris and Sih 1965; Wu and Router (1965); Lee and Advani (1982)). The special case *p=1*, *q=1* yields the mixed-mode criterion (Huang and Lin, 1996):

$$K_I / K_{Ic} + K_{II} / K_{IIc} = 1 \qquad (2)$$

The analysis of the empirical mixed-mode criteria for different values of *p* and *q*, shows that the most conservative criterion is given by equation (2) (Fig.1).

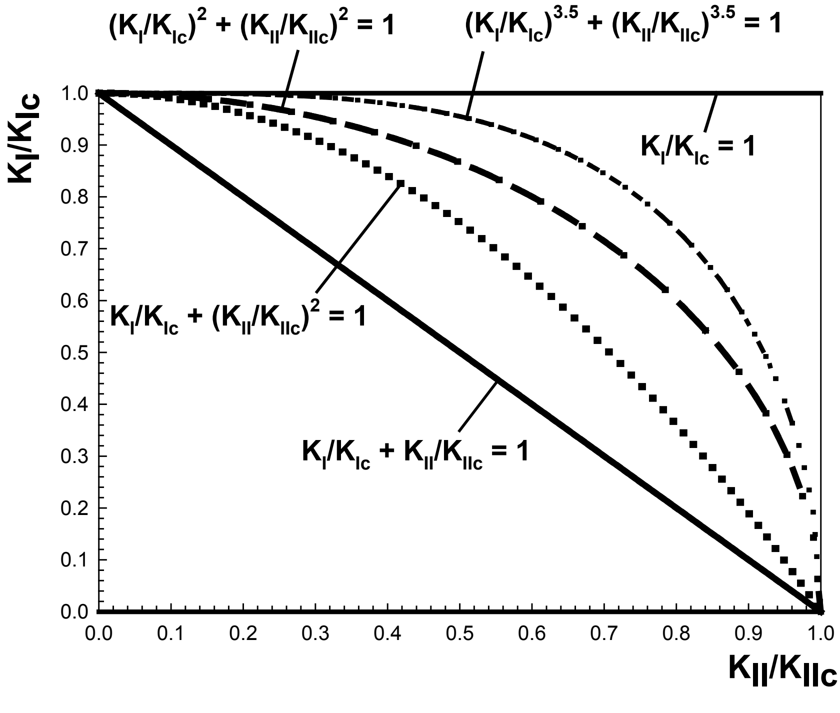

Figure 1. A comparison of the mixed mode failure criteria.

However, published experimental data show that pure mode I and mode II fracture toughness is associated with scatter (Wu and Router, 1965). As a result, equation (2), where K_{Ic} and K_{IIc} are average values, does not define a safe zone away from scatter associated with the combinations of stress intensity factors resulting in failure. For example, the average mode I fracture toughness is in the range ($K_{Ic,\min} \le K_{Ic} \le K_{Ic,\max}$) where $K_{Ic,\min}$ is the smallest and $K_{Ic,\max}$ is the largest fracture toughness measurement. Consequently, for a failure state characterised by a stress intensity factor $K_{II} = 0$ and a stress intensity factor K_I for which $K_{Ic,\min} \le K_I \le K_{Ic}$ holds, the combination of the stress intensity factors defines a failure state despite that $K_I / K_{Ic} + 0 / K_{IIc} < 1$. This means that failure criterion (2) cannot be used without a modification to define a safe zone. To define a true safe zone, the failure criterion (2) should be modified to

$$K_I / K_{Ic,\min} + K_{II} / K_{IIc,\min} = 1 \tag{3}$$

where $K_{Ic,\min}$ and $K_{IIc,\min}$ correspond to the smallest (not the average) measured fracture toughness values characterising mode I and mode II crack opening, respectively. The fracture criterion (3) will be referred to as *conservative empirical fracture criterion*.

To check the compliance of stress intensity factors defining fracture, with the conservative empirical criterion (3), experiments have been conducted with specimens from polymeric glass (Polymethylmethacrylate, PMMA) which fails in a brittle fashion. Two different batches of PMMA sheets were purchased from two different manufacturers: the first batch included sheets with thickness 5mm and the second batch included sheets with thickness 4.8mm. The shape of the specimens was of the type used in the experiments conducted by Ayatollahi and Aliha (2009): squares with side *L*=150mm with pre-cracks inclined at different angles α. The pre-crack slots were inclined at 0, 22.5, 45, 55, 62.5 and 72.5 degrees and after their cutting a sharp crack tip was formed at the ends of the crack slots by taping a razor blade in the machined pre-crack slot.

The tests were conducted on a Testometric tensile test machine, where the load was applied through two cylindrical pins fitted in holes with diameters 10mm, drilled in two opposite corners of each specimen (Fig.2).

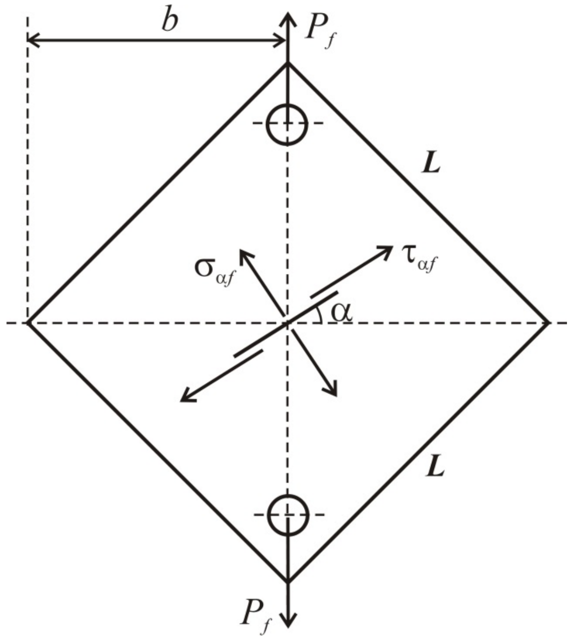

Figure 2. A test specimen used in the experimental study.

The maximum load P_f at which fracture occurred was recorded. The remote stress σ_f corresponding to fracture of the specimen and acting at the centre of the plate (in the absence of a crack) was estimated from a formula obtained by using the theory of elasticity:

$$\sigma_f \approx P_f/(bt) \tag{4}$$

where b is the half of the diagonal of the square plate and t is the thickness of the plate.

The normal stress $\sigma_{\alpha f}$ and the shear stress $\tau_{\alpha f}$ acting on the crack, at the point of fracture, were calculated from:

$$\sigma_{\alpha f} = \sigma_f \cos^2\alpha = \frac{P_f}{bt}\cos^2\alpha \tag{5}$$

$$\tau_{\alpha f} = \sigma_f \cos\alpha \sin\alpha = \frac{P_f}{bt}\cos\alpha \times \sin\alpha \tag{6}$$

The tensile opening mode stress intensity factor $K_{I\alpha}$ and the sliding opening mode stress intensity factor $K_{II\alpha}$ corresponding to an inclination angle α, at the point of specimen fracture and have been estimated from:

$$K_{I\alpha} = Y_I \sigma_{\alpha f} \sqrt{\pi a} \quad (7)$$

and

$$K_{II\alpha} = Y_{II} \tau_{\alpha f} \sqrt{\pi a} \quad (8)$$

where $Y_I \approx 1$, $Y_{II} \approx 1$; $\sigma_{\alpha f}$ and $\tau_{\alpha f}$ are given by equations (5) and (6), respectively.

For the first batch, from series of experiments, the lowest measured fracture toughness $K_{Ic,\min} = 0.991 \ MP\sqrt{m}$, corresponding to a tensile opening mode, and the lowest measured fracture toughness value $K_{IIc,\min} = 0.8 \ MP\sqrt{m}$, corresponding to the sliding crack opening mode were determined. The rest of the dimensionless fracture toughness values forming the scatter of the fracture toughness values were plotted as $K_{Ic,i} / K_{Ic,\min}$ and $K_{IIc,i} / K_{IIc,\min}$, for *i*=1,2,....(Fig.3).

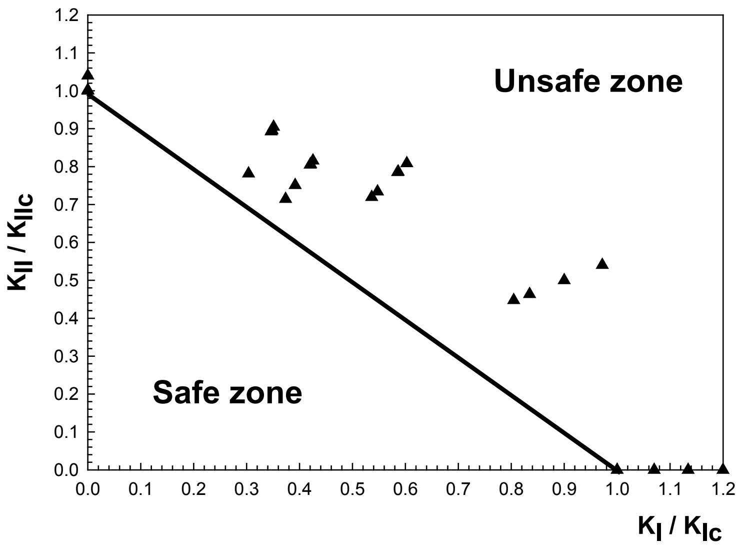

Figure 3. Locus of dimensionless stress intensity factors $K_{I\alpha} / K_{Ic}$, $K_{II\alpha} / K_{IIc}$ defining fracture, for the specimens from the first batch, for inclination angles 22.5°, 45°, 55° and 62.5°. The safe zone defined by the conservative empirical criterion (3) is away from the locus of stress intensity factors defining fracture.

No discussion has been found in the literature regarding the variation of the point of fracture as a function of $K_{I\alpha} / K_{Ic}$ and $K_{II\alpha} / K_{IIc}$ for a constant crack inclination angle α. Such variation however is clearly present, as can be verified from the experiments in (Fig.3). The locus of combinations of stress intensity factors (K_I / K_{Ic}, K_{II} / K_{IIc}) determining fracture is associated with uncertainty, even for such a homogeneous material as PMMA, for which the bluntness of the crack tip was carefully controlled. For an inhomogeneous material and a crack-like flaw with varying degree of bluntness, the expected uncertainty related to the locus of stress intensity factors defining failure at a specified orientation angle, will be even greater. Even if the fracture behaviour of a particular material follows a particular criterion (for example the maximum hoop stress criterion, (Erdogan and Sih, 1963)), because of the scatter at a crack inclination angle α, the locus of stress-intensity factors defining a fracture state *is not a line described by a simple function but a cloud of points forming an uncertainty band* (Fig.3).

To confirm the existence of such uncertainty band, a second series of experiments has been performed with a batch of PMMA samples with thickness 4.8mm, from another supplier. The smallest fracture toughness measurements characterising the material from the second supplier were $K_{Ic} = 0.924 \quad MP\sqrt{m}$ (corresponding to a tensile opening mode) and $K_{IIc} = 0.85$ (corresponding to a sliding opening mode). The dimensionless combinations $K_{I\alpha} / K_{Ic}$, $K_{II\alpha} / K_{IIc}$ defining fracture for each specimen from the second supplier have been plotted in Fig.4. They correspond to crack inclination angles of 22.5°, 45° and 55°. For the second batch, the lowest measured fracture toughness values were $K_{Ic,\min} = 0.940 \ MP\sqrt{m}$ (corresponding to a tensile opening mode) and $K_{IIc,\min} = 0.82 \ MP\sqrt{m}$ (corresponding to the sliding crack opening mode). Similar to the first batch, the rest of the dimensionless fracture toughness values forming the scatter of the fracture toughness values were plotted as $K_{Ic,i} / K_{Ic,\min}$ and $K_{IIc,i} / K_{IIc,\min}$, for *i*=1,2,....(Fig.4).

As it can be verified from the plot, the uncertainty related to the locus of stress intensity factors defining fracture, at a specified orientation angle, was also present for the material from the second supplier.

Despite that published experimental results involving inclined cracks clearly indicate the existence of an uncertainty band (e.g. Erdogan and Sih, 1963), many attempts have been made to fit a simple function to describe the locus of stress intensity factors defining fracture. However, the experiments clearly demonstrate that no such locus exists. The uncertainty associated with the locus of stress intensity factors defining fracture at a specified crack inclination angle, has not yet been appreciated in published experimental studies, invariably focussed on obtaining the parameters of a particular deterministic line fitting the locus of stress intensity factors defining fracture. In the presence of uncertainty in the locus of stress intensity factors defining fracture, the deterministic concept 'fracture criterion' given by a deterministic equation, cannot guarantee the safety of the designed components. To guarantee a low risk of failure for safety-critical components, a conservative safety zone needs to be specified, away from the scatter associated with the locus of stress intensity factors defining fracture. Such a conservative safe zone is provided by the failure criterion (3).

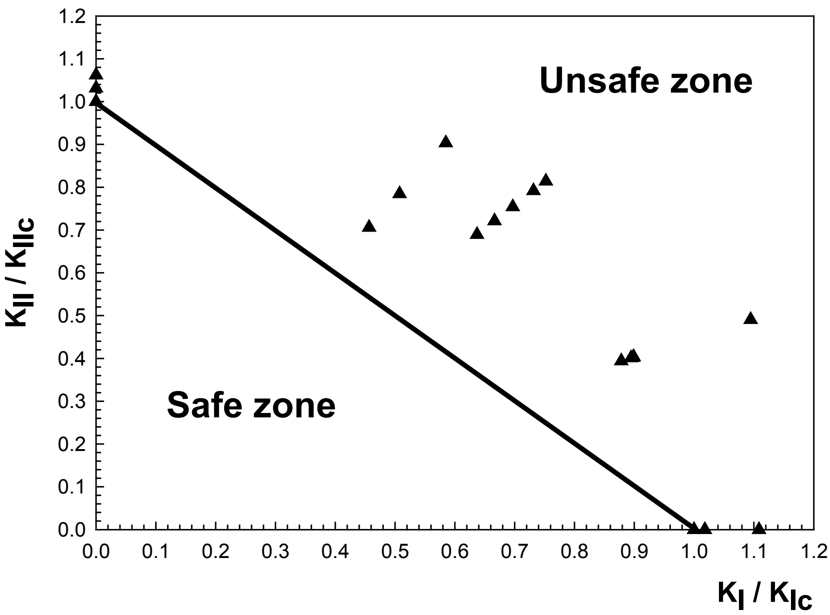

Figure 4. Locus of dimensionless stress intensity factors $K_{I\alpha}/K_{Ic}$, $K_{II\alpha}/K_{IIc}$ defining fracture for the specimens from the second batch. The inclination angles are 22.5°, 45° and 55°. The safe zone defined by equation (3) is away from the locus of stress intensity factors defining fracture.

3. A fracture condition incorporating the conservative safe zone and the most unfavourable orientation of the crack

3.1 Derivation of the condition

Selecting a conservative safe zone is not sufficient to test for a mixed –mode fracture. There is a need for a condition reflecting the most unfavourable orientation of the crack plane which corresponds to the most unfavourable combination of K_I and K_{II} stress intensity factors, providing the largest driving force for crack extension. The sharp penny-shaped crack is assumed to be with diameter equal to the detection threshold limit of the used inspection technique. For a penny-shaped crack and a tensile crack opening mode,

$$K_I = Y_I \sigma_n \sqrt{\pi a} , \qquad (9)$$

where a is half the crack diameter, σ_n is the stress normal to the crack plane and $Y_I = 2/\pi$ (Williams 1957; Anderson 1955). Similarly, for a sliding opening mode:

$$K_{II} = Y_{II} \tau \sqrt{\pi a} , \qquad (10)$$

where τ is the shear stress parallel to the crack plane and $Y_{II} = 1$. As a result, the conservative mixed-mode criterion (3) can be presented as

$$\sigma_n \theta + \tau \gamma \ge 1 \qquad (11)$$

where

$$\theta = Y_I \sqrt{\pi a} / K_{Ic} \qquad (12)$$

and

$$\gamma = Y_{II} \sqrt{\pi a} / K_{IIc} \qquad (13)$$

are material parameters depending on the size of the crack and the fracture toughness of the matrix characterising mode I and mode II crack opening.

For a sharp penny-shaped crack, with size equal to the detection threshold of the inspection technique, the most dangerous orientation is along a plane for which the expression $\sigma_n \theta + \tau \gamma$ attains a maximum. The orientation of the crack plane is given by the direction cosines $t_1 = \cos\alpha_1$, $t_2 = \cos\alpha_2$ and $t_3 = \cos\alpha_3$ ($t_1^2 + t_2^2 + t_3^2 = 1$), where α_1, α_2 and α_3 are the angles which the normal to the crack plane subtends with the coordinate axes x_1, x_2 and x_3 (Fig.5).

Consequently, finding the orientation for which the crack will be unstable, reduces to finding the orientation for which the expression

$$A(t_1, t_2) = \sigma_n \theta + \tau \gamma \tag{14}$$

has a maximum with respect to the direction cosines t_1, t_2 and t_3 followed by checking whether this maximum is equal to or greater than one.

$$\max_{t_1 t_2} A(t_1, t_2) \geq 1, \tag{15}$$

Expressing the normal and shear stress acting on the crack plane by the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and the direction cosines t_1, t_2 and t_3 of the crack plane normal (Fig.5), gives:

$$\sigma_n = t_1^2 \sigma_1 + t_2^2 \sigma_2 + t_3^2 \sigma_3 \tag{16}$$

$$\tau = [(\sigma_1 - \sigma_2)^2 t_1^2 t_2^2 + (\sigma_2 - \sigma_3)^2 t_2^2 t_3^2 + (\sigma_1 - \sigma_3)^2 t_1^2 t_3^2]^{1/2} \tag{17}$$

Therefore, in order to find $\max_{t_1, t_2} A(t_1, t_2)$ in equation (15), the function:

$$A(t_1, t_2) = [t_1^2 \sigma_1 + t_2^2 \sigma_2 + (1 - t_1^2 - t_2^2) \sigma_3] \theta +$$
$$+ [(\sigma_1 - \sigma_2)^2 t_1^2 t_2^2 + (\sigma_2 - \sigma_3)^2 t_2^2 (1 - t_1^2 - t_2^2) + (\sigma_1 - \sigma_3)^2 t_1^2 (1 - t_1^2 - t_2^2)]^{1/2} \gamma, \tag{18}$$

obtained by using equations (16) and (17), is to be maximised with respect to t_1 and t_2, in the closed domain $t_1^2 + t_2^2 \leq 1$.

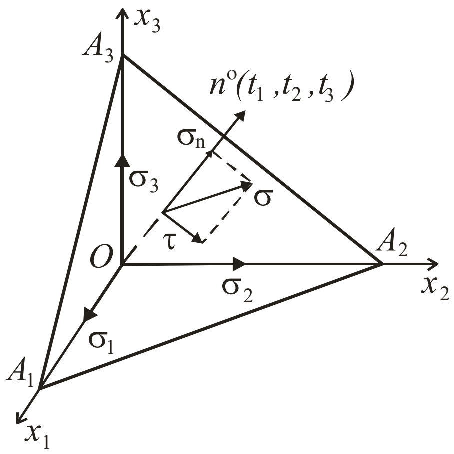

Figure 5. Normal and shear stress components acting on the potential crack plane $A_1A_2A_3$.

The local extrema of expression (18) can be found by using the necessary conditions

$$\frac{\partial A(t_1,t_2)}{\partial t_1}=0; \quad \frac{\partial A(t_1,t_2)}{\partial t_2}=0;$$

which lead to the non-linear system

$$\begin{aligned}&t_1[(\sigma_1-\sigma_3)t_1^2+(\sigma_2-\sigma_3)t_2^2-\frac{1}{2}(\sigma_1-\sigma_3)\\&-(\theta/\gamma)\,((\sigma_1-\sigma_2)^2t_1^2t_2^2+(\sigma_2-\sigma_3)^2t_2^2t_3^2+(\sigma_1-\sigma_3)^2t_1^2t_3^2)^{1/2}]=0\end{aligned} \tag{19}$$

$$\begin{aligned}&t_2[(\sigma_1-\sigma_3)t_1^2+(\sigma_2-\sigma_3)t_2^2-\frac{1}{2}(\sigma_2-\sigma_3)\\&-(\theta/\gamma)((\sigma_1-\sigma_2)^2t_1^2t_2^2+(\sigma_2-\sigma_3)^2t_2^2t_3^2+(\sigma_1-\sigma_3)^2t_1^2t_3^2)^{1/2}\,]=0\end{aligned} \tag{20}$$

with a trivial solution $t_1=t_2=0$, where $\theta=Y_I\sqrt{\pi a}/K_{Ic}$ and $\gamma=Y_{II}\sqrt{\pi a}/K_{IIc}$.

The stationary points of function (18), needed to determine the local maxima are among the solutions of system (19)-(20). The trivial solution is obtained from equation (18) by a direct substitution: $A(0,0)=\theta\sigma_3$.

Since there are no solutions of the system (19)-(20) for t_1 and t_2 both non-zero if $\sigma_1\neq\sigma_2$, the non-trivial solutions can be obtained by setting $t_1\neq 0, t_2=0$ and $t_1=0,\ t_2\neq 0$. After some complex algebraic manipulations (which, for the sake of readability will not be reproduced here), at the stationary points:

$$t_1 \equiv t_1^* = \pm\sqrt{\frac{1+\sqrt{\theta^2/(\theta^2+\gamma^2)}}{2}}\,,\ t_2 \equiv t_2^* = 0$$

two equal local maxima ($\max_{t_1,t_2} A(t_1,t_2) = A(t_1^*,t_2^*)$) of expression (18) are found, in the closed domain $t_1^2 + t_2^2 \leq 1$. Two distinct crack plane orientations correspond to these local maxima, with direction cosines of the crack plane normal given by:

$$t_1^* = \cos(\alpha_1) = \sqrt{\frac{1+\sqrt{\theta^2/(\theta^2+\gamma^2)}}{2}}\,,\ t_2^* = \cos(\alpha_2) = 0\,,$$

$$t_3^* = \cos(\alpha_3) = \sqrt{\frac{1-\sqrt{\theta^2/(\theta^2+\gamma^2)}}{2}} \qquad (21)$$

and

$$t_1^* = \cos(\alpha_1) = -\sqrt{\frac{1+\sqrt{\theta^2/(\theta^2+\gamma^2)}}{2}}\,,\ t_2^* = \cos\alpha_2) = 0\,,$$

$$t_3^* = \cos(\alpha_3) = \sqrt{\frac{1-\sqrt{\theta^2/(\theta^2+\gamma^2)}}{2}} \qquad (22)$$

The other combinations of the signs of t_1 and t_3 do not produce distinct crack planes. Substituting equations (21) into (18) results in:

$$\max_{t_1,t_2} A(t_1,t_2) = A(t_1^*,t_2^*) = \frac{\sigma_1+\sigma_3}{2}\theta + \frac{\sigma_1-\sigma_3}{2}\sqrt{\theta^2+\gamma^2} \qquad (23)$$

for the values of the local maxima. The global maximum of expression (18) is attained either at some of the local maxima given by the non-trivial solutions of the non-linear system (19)-(20) or at the boundary of the domain $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq \sqrt{1-t_1^2}$ ($t_1^2 + t_2^2 \leq 1$). Since t_1 and t_2 cannot be both nonzero, the boundary is defined by $t_1 = \pm 1$, $t_2 = 0$ and $t_1 = 0$, $t_2 = \pm 1$. Since $A(+1,0) = A(-1,0)$, $A(0,+1) = A(0,-1)$, the check of the function values on the boundary of the domain where t_1 and t_2 vary, reduces to a single check at points $t_1 = 1$,

$t_2 = 0$ and $t_1 = 0$, $t_2 = 1$ on the domain. The values $A(1,0)$ and $A(0,1)$ are obtained by a direct substitution in expression (18):

$$A(1,0) = \theta\sigma_1; \quad A(0,1) = \theta\sigma_2$$

The magnitudes of the principal stresses are in descending order ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) hence for $\theta > 0$, the inequalities $\theta\sigma_1 \geq \theta\sigma_2 \geq \theta\sigma_3$ hold. The global maximum of expression (18) is either equal to the local maximum (23) or to the value $A(1,0) = \theta\sigma_1$, whichever is the largest. By comparing the two values, we get

$$\frac{\sigma_1 + \sigma_3}{2}\theta + \frac{\sigma_1 - \sigma_3}{2}\sqrt{\theta^2 + \gamma^2} - \sigma_1\theta = (\sigma_1 - \sigma_3)\frac{\sqrt{\theta^2 + \gamma^2} - \theta}{2} > 0 \qquad (24)$$

because $\sigma_1 - \sigma_3 > 0$ and $\frac{\sqrt{\theta^2 + \gamma^2} - \theta}{2} > 0$.

Consequently, the global maximum in the closed domain $t_1^2 + t_2^2 \leq 1$ coincides with one of the local maxima, whose magnitude is given by expression (23). As a result, the condition for brittle fracture, incorporating the most unfavourable orientation of the crack plane, becomes:

$$\frac{\sigma_1 + \sigma_3}{2}\theta + \frac{\sigma_1 - \sigma_3}{2}\sqrt{\theta^2 + \gamma^2} \geq 1 \qquad (25)$$

The exact, most unfavourable orientation of the crack plane corresponding to the maximum driving force behind the crack extension is specified by equations (21)-(22) which give the direction cosines characterizing the normal to the crack plane.

The simple condition given by equation (25) incorporates the most unfavourable crack orientation and provides an answer to the central question whether a randomly oriented crack at a particular location will cause fracture.

3.2 Analysis of the new criterion

The correctness of the derived criterion can be confirmed by checking whether it models correctly the extreme cases. For a hydrostatic tension (stress state $\sigma_1 = \sigma_2 = \sigma_3 = +\sigma$), the maximum given by expression (23) becomes

$$\max_{t_1,t_2} A(t_1,t_2) = \theta\sigma$$

The direct substitution in equation (18) also results in $A(t_1,t_2) = \theta\sigma$. Indeed, in this case, shear stresses are absent, and the only crack opening mode is the tensile mode. Failure is initiated when $\theta\sigma = 1$, or when $Y_I \sqrt{\pi a} = K_{Ic}$, as it should be.

For materials characterised by a very small mode II fracture toughness and a large mode I fracture toughness $\gamma >> \theta$ and if θ is neglected, equations (19) and (20) result in: $t_1^* = \pm\sqrt{1/2},\ t_2^* = 0,\ t_3^* = \pm\sqrt{1/2}$. For the maximum of expression (23), we get:

$$\max_{t_1,t_2} A(t_1,t_2) = A(t_1^*,t_2^*) = \frac{\sigma_1 - \sigma_3}{2}\gamma = \tau_{\max}\gamma$$

In this case, the plane of the crack with the most unfavourable orientation coincides with the maximum shear stress τ_{max}, acting at an angle of 45° with respect to σ_1 and σ_3. Failure will be initiated when $\tau_{\max}\gamma = 1$, or when $Y_{II} \sqrt{\pi a} = K_{IIc}$, as it should be.

Finally, given particular values of the parameters θ and γ, the maximum in (25) is attained if $\sigma_1 > 0$ and $\sigma_3 < 0$. This can be seen immediately if the fracture condition (25) is re-arranged as

$$\theta\left(\frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sqrt{1 + \gamma^2/\theta^2}\right) \geq 1 \qquad (26)$$

and noticing that $\sqrt{1 + \gamma^2/\theta^2} \geq 1$ always holds. For $\sigma_1 > 0$ and $\sigma_3 > 0$, the value of the expression on the left hand side of equation (26) is smaller than the value of the same expression for $\sigma_1 > 0$ and $\sigma_3 < 0$. These analyses are in agreement with the experimental observations. Consequently, the derived condition models correctly the extreme cases.

The correctness of the derived condition will be verified by using the two-dimensional stress state from Section 2, for which there is exact analytical solution. For the crack in Fig.2, for the sake of simplicity, $Y_I = 1$, $Y_{II} = 1$, $K_{Ic,\min} = K_{IIc,\min} = K_c$ is assumed. The fracture criterion (3) then becomes:

$$\frac{\sigma_f \sqrt{\pi a}}{K_c}(\cos^2\alpha + \sin\alpha \times \cos\alpha) > 1 \tag{27}$$

The maximum of the expression

$$f(\alpha) = \cos^2\alpha + \sin\alpha \times \cos\alpha \tag{28}$$

is found at the value of α for which $\frac{df(\alpha)}{d\alpha} = 0$ and $\frac{d^2 f(\alpha)}{d\alpha^2} < 0$. After differentiation and some algebra, the value of α for which $f(\alpha)$ has a maximum is determined from

$$\tan 2\alpha = 1, \tag{29}$$

from which $\alpha = 22.5^\circ$. The value of the maximum is $f(22.5^\circ) = 1.207$ and inequality (27) becomes:

$$1.207 \times \frac{\sigma_f \sqrt{\pi a}}{K_c} > 1 \tag{30}$$

According to the proposed condition incorporating the most unfavourable orientation of the crack, the maximum of expression (18) is obtained for a direction cosine ($\cos\alpha$)

$$t_1^* \equiv \cos\alpha = \sqrt{\frac{1 + \sqrt{\theta^2/(\theta^2 + \gamma^2)}}{2}} \tag{31}$$

Because of the assumption $Y_I = 1$, $Y_{II} = 1$, $K_{Ic,\min} = K_{IIc,\min} = K_c$, it follows that $\theta = \gamma$ and expression (31) becomes

$$\cos\alpha = \sqrt{\frac{1 + \sqrt{1/2}}{2}} = 0.92388, \tag{32}$$

from which $\alpha = 22.5^\circ$.

Because the principal stresses are $\sigma_1 = \sigma_f$ and $\sigma_3 = 0$ and $\theta = \gamma$, the maximum of the fracture condition (25) becomes

$$\max_{t_1 t_2} A(t_1, t_2) = \sigma_f \theta [1/2 + \sqrt{2}/2] = 1.207 \times \frac{\sigma_f \sqrt{\pi a}}{K_c} \tag{33}$$

As a result, the analytical result for the most unfavourable orientation of a crack in a 2D stress field and the result from the derived condition coincide. This constitutes a validation of the proposed fracture condition incorporating the most unfavourable crack orientation for the 2D stress state.

In a general three-dimensional stress state however, the correctness of the derived condition must be verified through a specially designed computer program which determines the maximum of expression (18) directly. Accordingly, a program determining directly the global maximum of expression (18) has been designed. Here is a test example assuming material characterised by $K_{Ic} = 45\ MPa\sqrt{m}$ and $K_{IIc} = 31.5\ MPa\sqrt{m}$. A globular flaw with diameter $2a=600\ \mu m$ has been assumed, from which emanates a penny-shaped crack with the most unfavourable orientation. The principal stresses characterising the flaw location are $\sigma_1 = 1400\ MPa$, $\sigma_2 = 300\ MPa$ and $\sigma_3 = -510\ MPa$.

The calculated numerical values for the constants θ and γ are $\theta \approx 0.43 \times 10^{-9}\ MPa^{-1}$ and $\gamma \approx 0.97 \times 10^{-9}\ MPa^{-2}$. For the shape factor Y_{II} of the sliding opening mode, $Y_{II} \approx 1$ has been assumed.

A global maximum max $A(t_1, t_2) = 1.2$ of expression (18) attained at $t_1^* = \pm 0.838$, $t_2 = 0$ was found by the specially developed program. These values were confirmed by substituting the numerical values of the parameters in the closed-form solution (25). Agreement with the theoretical solution was obtained for various combinations of the principal stresses.

The proposed fracture condition for mixed-mode brittle facture can be applied to check the safety of loaded brittle components with complex shape, where fracture is locally initiated by flaws. Ceramics, high-strength steels, glasses, stone, etc., are examples of materials with such a failure mechanism. The described model is also valid for components from low carbon steels undergoing cleavage fracture at low temperature. Cleavage in steels usually propagates from cracked inclusions (Rosenfield, 1997; McMahon and Cohen, 1965). It usually involves a small amount of local plastic deformation to produce dislocation pile-ups and crack initiation from a particle which has cracked during the plastic deformation.

A postprocessor based on the new fracture condition can be easily developed, for testing loaded safety-critical components with complex shape. For each finite element, only a single computation of the fracture criterion is made. This guarantees a high overall computational speed, which makes the postprocessor particularly suitable for testing safety-critical designs in a design optimisation loop.

The prosed fracture condition is particularly suitable for optimising the shape of brittle components (for example the cross section of ceramic beams), in order to increase the resistance to failure locally initiated by flaws.

Conclusions

A fracture condition incorporating the most unfavourable crack orientation condition has been proposed to improve the safety of loaded brittle components with complex shape, whose loading results in a three- dimensional stress state. With a single calculation, the proposed fracture condition provides an answer to the important question whether a randomly oriented crack at a particular location in the stressed component will cause failure.

2. The proposed fracture condition has a relatively simple analytical form and can be used as a convenient tool for ensuring a conservative design for brittle components.

3. The proposed fracture condition makes it unnecessary to test for crack instability along possible random orientations of the crack at a given location. As a result, the attained computational speed is orders of magnitudes higher than the computational speed characterisng the direct approach.

5. The fracture locations defined by the combinations of dimensionless stress intensity factors $K_{I\theta}/K_{Ic}$, $K_{II\theta}/K_{IIc}$ are subjected to a great deal of uncertainty at any specified inclination of the crack. Deterministic functions used to describe the locus of stress intensity factors defining failure state lead to unsafe designs. A conservative safe zone, located away from the scatter band defining fracture states should be used instead.

6. Fracture criteria based on dimensionless stress intensity factors $K_{I\theta}/K_{Ic}$, $K_{II\theta}/K_{IIc}$, where K_{Ic} and K_{IIc} is the average fracture toughness characterising mode I and mode II crack opening mode should not be used in the fracture criterion $K_I/K_{Ic} + K_{II}/K_{IIc} = 1$ to define a safe zone. The smallest values of the fracture toughness values should be used instead.

7. A postprocessor based on the proposed fracture condition can be easily developed, for testing loaded safety-critical components with complex shape. For each finite element, only a single computation is made, which guarantees a high overall computational speed. This makes the proposed approach particularly useful for incorporation in a design optimisation loop.

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