Generation of new knowledge and optimisation of systems and processes through meaningful interpretation of sub-additive functions

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Abstract: The paper introduces a method for increasing the impact of extensive quantities by meaningful interpretation of multivariate sub-additive and super-additive functions. The paper demonstrates that the segmentation of extensive quantities through sub-additive and super-additive functions can be used to generate new knowledge and optimise systems and processes and the presented algebraic inequalities are applicable to any area of science and technology.

The meaningful interpretation of the modified Cauchy-Schwarz inequality, led to a method for increasing of the power output from a voltage source and to a method for increasing the capacity for absorbing strain energy of loaded mechanical components. It was found that the existence of asymmetry is essential to increasing the strain energy absorbing capacity and the power output. Loaded elements experiencing the same displacement do not yield an increase of the absorbed strain energy. Similarly, loaded resistances experiencing the same current do not yield an increase of the power output.

Finally, the meaningful interpretation of an algebraic inequality in terms of potential energy, resulted in a general necessary condition for minimising the sum of powers of distances to a fixed number of points in space.

Keywords: algebraic inequalities; sub-additive multivariate functions, super-additive multivariate functions; extensive quantities, segmentation

1 Introduction

Algebraic inequalities have been used extensively in mathematics and a number of useful non-trivial algebraic inequalities and their properties have been well documented (Bechenbach and Bellman, 1961; Cloud et al, 1998; Engel, 1998; Hardy, 1999; Pashpatte, 2005; Steele, 2004; Kazarinoff, 1961; Sedrakyan and Sedrakyan, 2010; Belabess, 2019).

A comprehensive overview of the use of inequalities in mathematics has been presented in (Fink, 2000). For a long time, simple inequalities are being used to express error bounds in approximations and constraints in linear programming models.

Applications of inequalities have been considered in physics (Rastegin, 2012) and engineering (Cloud et al, 1998; Samuel and Weir, 1999). In engineering design, design inequalities have been widely used to express design constraints guaranteeing that the design will perform its required function. This paper shows that the use of algebraic inequalities in engineering is far reaching and is certainly not restricted to specifying design constraints. The meaningful interpretation is a new dimension in the use of algebraic inequalities.

In reliability and risk research, inequalities have been used as a tool for characterisation of reliability functions (Ebeling, 1997; Xie and Lai, 1998; Makri and Psillakis, 1996; Hill et al., 2013; Berg and Kesten, 1985; Kundu and Ghosh, 2017; Dohmen, 2006).

Algebraic inequalities are particularly suitable for handling uncertainty. The conventional approaches for handling uncertainty is through probabilities assessed by using a data-driven approach or by using the Bayessian, subjective probability approach. A major deficiency of the data-driven approach is that probabilities cannot always be meaningfully defined. Thus, models based on failure rate data collected for a particular environment (temperature, humidity, pressure, vibrations, corrosive agents, etc.) give poor predictions for the time to failure in a different environment. In addition, to make any predictions, the data-driven approach is critically dependent on the availability of past failure rates.

The deficiencies of the data-driven approach cannot be rectified by using the Bayessian approach which is not so critically dependent on the availability of past failure data since it uses assigned subjective probabilities. The Bayessian approach however, depends on a selected probability model that may not be relevant to the modelled phenomenon/process (Aven, 2017). In addition, the assigned subjective probabilities depend on the available knowledge and vary significantly among the assessors. Furthermore, weak background knowledge at the basis of the assigned subjective probabilities often results in poor predictions.

In using inequalities to handle uncertainty, there is no need to assign frequentist probabilities, subjective probabilities or any particular probabilistic model. Algebraic inequalities *do not require knowledge related to the distributions of the variables entering the inequalities* and this makes a method based on algebraic inequalities ideal for handling deep uncertainty associated with components, properties and control parameters.

Despite that the probabilities remain unknown, the algebraic inequalities can still establish the intrinsic superiority of one of the competing options. In this respect, algebraic inequalities avoid the major drawback in the conventional models for handling uncertainty – poor predictions due to lack of meaningful specification of frequentist probabilities or weak knowledge behind assigned subjective probabilities and probabilistic models.

There are two major approaches of using algebraic inequalities for enhancing systems performance: (i) *a forward approach*, consisting of deriving an abstract algebraic inequality from a real physical system or process which is subsequently tested and proved rigorously and (ii) *an inverse approach*, consisting of interpreting a correct abstract inequality and inferring from it unknown properties related to a real physical system or process.

The forward approach includes several basic steps: (i) analysis of the system or process; (ii) conjecturing inequalities ranking the competing alternatives; (iii) testing the conjectured inequalities and (iv) proving the conjectured inequalities rigorously.

By following this approach, inequalities can be used to *rank systems with unknown reliabilities of their components.* The forward approach of exploiting algebraic inequalities has been demonstrated in (Todinov, 2018,2020).

Often, the conjectured inequalities hold (one of the systems/processes is superior to the other systems/processes) for some set of values for the controlling variables but for another set of values the conjectured inequalities do not hold. As a result, no intrinsically more reliable system or process emerges as a result of the application of the forward approach. Furthermore, the potential for generating new knowledge of the forward approach is confined only to specific performance characteristics, specific systems and processes.

These limitations can be overcome by the inverse approach. This approach, proposed in the paper, is based on the observation that abstract inequalities contain useful quantitative knowledge that can be *released by their meaningful interpretation*. Furthermore, depending on the specific interpretation, knowledge applicable to different systems from different domains *can be released from the same inequality*. In this sense, the inverse approach *does not require or imply any forward analysis* of pre-existing systems or processes. The real systems or processes to which the inequality applies are a result of the meaningful interpretation of the variables and the different parts of the inequalities.

The key step of the inverse approach is creating, relevant meaning for the variables entering a correct algebraic inequality, followed by a meaningful interpretation of the different parts of the inequality

Unlike the forward approach, the inverse approach always leads to new results, as long as meaningful interpretation of the algebraic inequality can be provided. This is because the inverse approach is rooted in the principle of non-contradiction: *if the variables and the different parts of a correct abstract inequality can be interpreted as a system or process, in the real world, the system or process exhibit properties that are consistent with the abstract inequality.* In other words, the realization of the process/experiment yields results that do not contradict the algebraic inequality.

The inverse approach effectively links existing correct abstract algebraic inequalities with real physical systems or processes and not only opens opportunities for enhanced performance of systems and processes but also leads to the discovery of new fundamental properties.

The key parts of the process of meaningful interpretation of an algebraic inequality are: (i) identifying classes of abstract inequalities which permit a meaningful interpretation that can be linked with a real system or process in various specific domains and (ii) identifying various transformations of known inequalities which make their meaningful interpretation possible in the listed domains.

Another contribution of this paper is the idea that sub-additive and super-additive multivariate functions can be used to enhance the impact of extensive quantities in all areas of science and technology by segmenting or aggregating extensive quantities.

The meaningful interpretation of these abstract inequalities helped to find overlooked properties in such mature fields like mechanical engineering and electrical engineering.

By using the derived new properties in the area of mechanical engineering, design engineers are able to maximise the strain energy accumulation capacity of various mechanical assemblies during impact loading. In the area of electrical engineering, practitioners are able to maximise the power output for example, for low temperatue heating.

Very important candidates for meaningful interpretation is the wide class of inequalities based on super-additive and sub-additive functions (Rosenbaum, 1950; Alsina and Nelson, 2020). Thus, if a particular function measures the effect of a particular factor, the sub-additive and the super-additive inequalities have the potential to significantly increase the effect of extensive quantities by segmenting them or by aggregating them.

1.2 Extensive and intensive quantities

Extensive quantities change with changing the size of their supporting objects/systems (DeVoe, 2012; Mannaerts, 2014) and can be found in all areas of science and engineering. DeVoe (2012), for example, introduced 'extensivity' test that consists of dividing the system by an imaginary surface into two parts. Any quantity characterising the system that is the sum of the same quantity characterising the two parts is an extensive quantity and any quantity that has the same value in each part of the system is an intensive quantity.

In the present treatment, extensivity will be approached through additivity. Additivity, for example, is present for mass, weight, amount of substance, number of particles, volume, distance, energy (kinetic energy, gravitational energy, electric energy, elastic energy, surface energy, internal energy), work, power, heat, force, momentum, electric charge, electric current, heat capacity, electric capacity, resistance (when the elements are in series), enthalpy, fluid flow. These are also examples of extensive quantities.

Intensive quantities characterise the object/system locally and do not change with changing the size of the supporting objects/systems. Additivity is not present for intensive quantities. Additivity, for example, is not present for pressure or temperature. Consider for example, a pressure vessel containing gas at a particular pressure and temperature. If a notional division of the pressure vessel is made, the temperature or pressure measured in the pressure vessel is not a sum of the temperature/pressure measured in the different parts of the vessel. Other properties where additivity is not present are 'density', 'concentration', 'hardness', 'velocity', 'acceleration', 'stress', 'surface tension', etc.

It is important to note that proportionality to mass is not a necessary condition for additivity. For a large group of extensive quantities (area, work, electric energy, elastic energy, displacement energy, surface energy, electric current, power, heat, electric charge), the proportionality to mass is missing (Mannaerts, 2014).

Depending on how the elements are arranged, additivity may be present or absent. Thus, for resistances arranged in series, additivity is present, for the equivalent resistance of elements connected in series is a sum of the individual resistances. For resistances arranged in parallel, additivity is absent.

Similarly, for voltage sources connected in series, additivity is present, for the total voltage is a sum of the voltages of the individual sources. Additivity is also present for capacitors connected in parallel. For this arrangement, the equivalent capacitance is a sum of the individual capacitances. For capacitors arranged in series, additivity is absent.

2 Sub-additive and super-additive multivariate functions

Multivariate sub-additive functions have been discussed in Rosenbaum (1950).
In the Euclidean space of one dimension, the sub-additive function $f(x)$ satisfies the inequality

$$f(x_1 + x_2) \le f(x_1) + f(x_2) \tag{1}$$

for any pair of points x_1 and x_2 in the domain of definition.
If the direction of the inequality is reversed, the function $f(x)$ is super-additive:

$$f(x_1 + x_2) \ge f(x_1) + f(x_2) \tag{2}$$

In the Euclidean space of two dimensions, the multivariate sub-additive function $f(x, y)$ satisfies the inequality

$$f(x_1 + x_2,\ y_1 + y_2) \le f(x_1, y_1) + f(x_2, y_2) \tag{3}$$

for any pair of points (x_1, y_1) and (x_2, y_2) in the domain of definition.
If the direction of the inequality is reversed, the function $f(x, y)$ is super-additive:

$$f(x_1 + x_2,\ y_1 + y_2) \ge f(x_1, y_1) + f(x_2, y_2) \tag{4}$$

Consider *n* points with positive coordinates x_i in the definition domain of the function *f()*. From definition (1), it follows that

$$f(x_1 + x_2 + ... + x_n) \le f(x_1) + f(x_2) + ... + f(x_n) \tag{5}$$

while from definition (2), it follows that

$$f(x_1 + x_2 + ... + x_n) \ge f(x_1) + f(x_2) + ... + f(x_n) \tag{6}$$

Key results related to sub-additive and super-additive functions of a single variable have been stated in (Alsina and Nelson, 2010). Thus, if a function $f()$, with a domain $[0, \infty)$ and range $[0,\infty)$ is concave, and $f(0)\geq 0$ then the function is sub-additive: $f(x_1+x_2+...+x_n)\leq f(x_1)+f(x_2)+...+f(x_n)$. If the function $f()$ is convex and $f(0)\leq 0$, then it is super-additive: $f(x_1+x_2+...+x_n)\geq f(x_1)+f(x_2)+...+f(x_n)$.

Consider now *n* pair of points (a_i,b_i) with positive coordinates a_i,b_i from the definition domain of the function *f()*. From definition (3), it follows that

$$f(a_1+a_2+...+a_n,\ b_1+b_2+...+b_n)\leq f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n) \quad (7)$$

while from definition (4) it follows that

$$f(a_1+a_2+...+a_n,\ b_1+b_2+...+b_n)\geq f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n) \quad (8)$$

Relationships (5)-(8) can be obtained by mathematical induction and, for the sake of brevity, details have been omitted.

Inequalities (5)-(8) have a number of powerful potential applications in increasing/decreasing the effect of extensive quantities (factors).

Inequalities (5) and (6), for example, can be applied for optimising processes. If the function $f(\bullet)$ reflects the effect/output of a particular factor and x_i denote the different segments of the factor, inequalities (5) and (6) provide the unique opportunity to increase the effect of the factor by segmenting it or aggregating it, depending on whether the function $f()$ is concave or convex. If the function is concave, with a domain $[0,\infty)$ and range $[0,\infty)$, segmenting the factor results in a larger output. Conversely, if the function is convex in this domain, aggregating the factor results in a larger output.

An important condition for using inequalities (5) and (6) is the outputs $f(x_1), f(x_2),..., f(x_n)$ after segmenting the control factor *x* to be additive. This is fulfilled if the outputs $f(x_1), f(x_2),..., f(x_n)$ are extensive quantities: energy, power, force, damage, profit, pollution, mass, number, volume, etc.

In order to apply inequalities (7) or (8), a_i, b_i and the quantities $f(a_i,b_i)$ must all be extensive quantities. Inequality (7), effectively states that the effect of the extensive quantities $A=\sum_{i=1}^{n}a_i$ and $B=\sum_{i=1}^{n}b_i$ can be increased by segmenting them into smaller parts a_i and b_i, $i=1,...,n$ and accumulating their individual effects $f(a_i,b_i)$ (represented by the sum of the terms on the right-hand side of inequality (7). Similarly, inequality (8) effectively states that the effect of the extensive quantities $A=\sum_{i=1}^{n}a_i$ and $B=\sum_{i=1}^{n}b_i$ can be decreased by segmenting them into smaller parts a_i and b_i, $i=1,...,n$ and accumulating their individual effects $f(a_i,b_i)$ (represented by the sum of the terms on the right-hand side of inequality (8)).

Inequalities (7) and (8) have a universal application in science and technology as long as a_i, b_i and the terms $f(a_i,b_i)$ are extensive quantities and have meaningful interpretation.

Inequalities similar to (7) and (8) can be derived for any number of factors by using the definition of sub-additive/super-additive functions. For example, for three factors *a,b* and *c*, the next inequalities are obtained:

$$f(a_1+...+a_n,\ b_1+...+b_n, c_1+...+c_n)\leq f(a_1,b_1,c_1)+f(a_2,b_2,c_2)+...+f(a_n,b_n,c_n) \quad (9)$$

for a sub-additive function $f(a,b,c)$ and

$$f(a_1+...+a_n,\ b_1+...+b_n, c_1+...+c_n)\geq f(a_1,b_1,c_1)+f(a_2,b_2,c_2)+...+f(a_n,b_n,c_n) \quad (10)$$

for a super-additive function $f(a,b,c)$.

3 Meaningful interpretation of a multivariate sub-additive function to maximise electric power output

A special case of the general sub-additive function (7) is the inequality

$$\frac{(a_1+a_2+...+a_n)^2}{b_1+b_2+...+b_n} \le \frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+...+\frac{a_n^2}{b_n} \tag{11}$$

valid for any sequence $a_1, a_2, ..., a_n$ of real numbers and any sequence $b_1, b_2, ..., b_n$ of positive real numbers. The role of the sub-additive function $f(a,b)$ in inequality (7) is played by the function $f(a,b) \equiv a^2/b$ in inequality (11).

It can be shown that inequality (11) is a transformation of the well-known Cauchy-Schwarz inequality (Steele, 2004) that states that for any two sequences of real numbers $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$, the following inequality holds:

$$(x_1y_1+x_2y_2+...+x_ny_n)^2 \le (x_1^2+x_2^2+...+x_n^2)(y_1^2+y_2^2+...+y_n^2) \tag{12}$$

Indeed, equality is present if and only if for any $i \ne j$, $x_iy_j = x_jy_i$ are fulfilled.

If the substitutions $x_i = \frac{a_i}{\sqrt{b_i}}$ $(i=1,...,n)$ and $y_i = \sqrt{b_i}$ $(i=1,...,n)$ are made in the Cauchy-Schwarz inequality (12) the result is the inequality

$$\left(\frac{a_1}{\sqrt{b_1}}\sqrt{b_1}+\frac{a_2}{\sqrt{b_2}}\sqrt{b_2}+...+\frac{a_n}{\sqrt{b_n}}\sqrt{b_n}\right)^2 \le \left((a_1/\sqrt{b_1})^2+...+(a_n/\sqrt{b_n})^2\right)\left((\sqrt{b_1})^2+...+(\sqrt{b_n})^2\right)$$

which leads to inequality (11), valid for any sequences $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ of positive real numbers. Equality in (11) is attained only if $a_1/b_1 = a_2/b_2 = ... = a_n/b_n$.

Inequality (11) has a number of interesting potential applications in increasing the effect of extensive quantities (factors).

In order to apply inequality (11), a_i, b_i and the ratios a_i^2/b_i must all be extensive quantities. Inequality (11) effectively states that the effect of the extensive quantities $A=\sum_{i=1}^{n} a_i$ and $B=\sum_{i=1}^{n} b_i$ can be increased by segmenting them into smaller parts a_i and b_i, $i=1,...,n$ and accumulating their individual effects a_i^2/b_i (represented by the sum of the terms on the right-hand side of inequality (11)).

Inequality (11) has a universal application in science and technology as long as a_i, b_i and the terms a_i^2/b_i are extensive quantities and have meaningful interpretation. The condition for applying inequality (11) is the possibility to present an extensive quantity p_i as a ratio of a square of extensive quantity a_i and another extensive quantity b_i:

$$p_i = a_i^2/b_i \tag{13}$$

As an example, consider a case where factor a is 'voltage' from a source whose elements are arranged in series (extensive quantity) and factor b is 'resistance' of elements arranged in series which is also extensive quantity. Suppose that the source of voltage V is applied to n elements in series, with resistances $r_1, r_2, ..., r_n$ (Figure 1a).

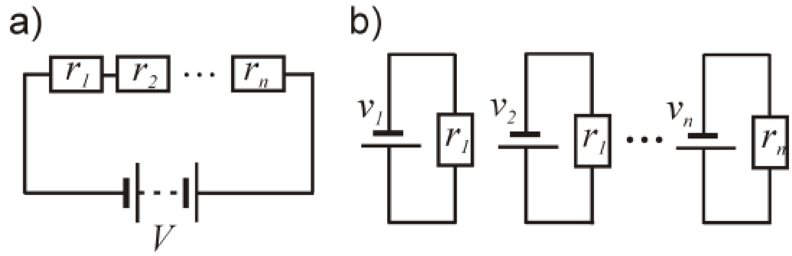
Consider segmenting the source of voltage V into n smaller sources of voltage v_i such that $V = v_1 + ... + v_n$ (Figure 1b).

Let $a_i = v_i$, $i = 1,...,n$ be the voltages applied to n separate elements with resistances r_i (Figure 1b). If $b_i = r_i$, the condition (13) is fulfilled because the equation

$$p_i = v_i^2 / r_i \quad (14)$$

holds, where p_i is the power output [W] created by the ith voltage source, v_i is the voltage and r_i is the resistance of the ith element. In equation (14), all variables v_i, p_i and r_i are extensive quantities.

Figure 1. a) A single voltage source applied to elements connected in series; b) A voltage source V segmented into n smaller sources v_i applied to the individual elements.



Inequality (11) can be rewritten as

$$\frac{V^2}{r_1 + r_2 + ... + r_n} \le \frac{v_1^2}{r_1} + \frac{v_2^2}{r_2} + ... + \frac{v_n^2}{r_n} \quad (15)$$

Both sides of inequality (15) can be meaningfully interpreted in the next result, relevant to electrical circuits:

The power output from a source with voltage V, on elements connected in series, never exceeds the total power output from the sources v_i ($\sum_i v_i = V$), *resulting from segmenting the original source V and applying the segmented sources to the individual elements.*

This result holds irrespective of the individual resistances r_i of the elements and the voltage sources v_i into which the original voltage V has been segmented.

Note that the existence of asymmetry in the system is essential for increasing the power output through segmentation of the voltage source. No increase in the power output is present if

$$v_1 / r_1 = v_2 / r_2 = ... = v_n / r_n = i \quad (16)$$

is fulfilled.

This means that combinations of resistances and sources resulting in the same current i do not yield an increase of the power output.

To maximise the right-hand side of inequality (15), the squared voltages arranged in descending order $v_1^2 \ge v_2^2 \ge ... \ge v_n^2$ must correspond (be applied) to the resistances arranged in ascending order ($r_1 \le r_2 \le ... \le r_n$). For resistances arranged in ascending order, their reciprocals are arranged in descending order ($1/r_1 \ge 1/r_2 \ge ... \ge 1/r_n$) and according to the rearrangement inequality (Steele, 2004), the dot product $v_1^2(1/r_1) + v_2^2(1/r_2) + ... + v_n^2(1/r_n)$ of two similarly ordered sequences is a maximum.

Thus, for resistances $r_1 = 10\Omega$, $r_2 = 15\Omega$, $r_3 = 25\Omega$, $r_4 = 50\Omega$ and voltage source of 16V, segmented into $v_1 = 6V$, $v_2 = 5V$, $v_3 = 3V$ and $v_4 = 2V$, the maximum possible power output is $P_{\max} = v_1^2 / r_1 + v_2^2 / r_2 + v_3^2 / r_3 + v_4^2 / r_4 = 5.7W$. Any other permutation of voltages and resistances results in a smaller power output.
If the voltage source $V = 16V$ is not segmented, applying the voltage V to the four elements in series delivers output power $P = V^2 / (r_1 + r_2 + r_3 + r_4) = 2.56W$, which is less than half the maximum power of $P_{\max} = 5.7W$ delivered in the case of a voltage source segmentation.

The result from the meaningful interpretation of the algebraic inequality can be applied in electric circuits for low-temperature heating. Despite that the circuits in Figure 1a and 1b seem to be very different electric systems, they both can be viewed as alternative designs of a direct heating system. The circuit in Figure 1b dissipates more power (heat) for the same total number of voltage elements and the same set of resistive elements. It needs to be pointed out that this advantage is not present if the segmented circuits are characterised by the same currents through the resistive elements.

Interestingly, no such property has been reported in modern comprehensive texts in the field of electronics (Floyd and Buchla, 2014; Horowiz and Hill, 2015). This demonstrates that the meaningful interpretation of the abstract inequality (11) helped to find an overlooked property even in such a mature field like electronics.

4. Segmentation of a force and area to increase the accumulated strain energy

An alternative meaning can be created for inequality (11) if variable a, for example, stands for the extensive quantity 'force' and variable b stands for the extensive quantity 'area'.

It is a well-known result from mechanics of materials (Hearn, 1985) that the accumulated strain energy U of a linearly elastic bar with length L and cross-sectional area A is given by the equation:

$$U = \frac{P^2 L}{2EA} \tag{17}$$

where E [Pa] denotes the Young modulus of the material and P [N] is the magnitude of the loading force. The strain energy U [J] is an extensive quantity.
Equation (17) can be written as

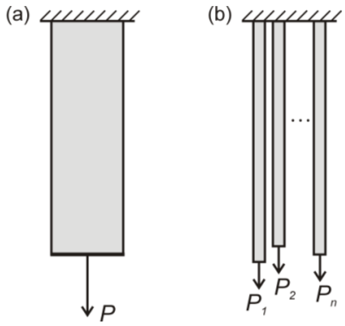
$$U = \frac{P^2}{2k} \tag{18}$$

where $k = EA / L$ [N/m] is the stiffness of the bar.

Consider the two designs in Figure 2. In the design configuration in Figure 2a, a single force P acts on a single large bar with cross-sectional area A. In the design configuration in Figure 2b, the original bar has been segmented into n individual bars with smaller cross sections $A_1, A_2, ..., A_n$, the sum of which is equal to the cross section A of the initial bar ($A = A_1 + ... + A_n$) and the force P has also been segmented into n forces $P_1, P_2, ..., P_n$ the sum of which is equal to the initial loading force P ($P = P_1 + P_2 + ... + P_n$). The smaller forces P_i have been applied to the individual bars independently, and the individual bars do not necessarily have the same displacements and stresses (Figure 2b). The support is rigid and does not deform due to the segmented forces.

A question of interest is which design configuration in Figure 2 is capable of accumulating a higher amount of elastic strain energy.

Figure 2. (a) A force *P* applied to a single bar; (b) Segmented forces P_i, applied to individual bars into which the original bar has been segmented (the individual bars are loaded independently and they do not necessarily have the same displacements and stresses)



Let $a_i = P_i$, $i = 1,...,n$ be the forces applied to the centroids of the *n* separate bars whose stiffness values are $k_i = E_i A_i / L_i$ (Figure 2b).

The systems in Figure 2a and 2b are not equivalent mechanically and the equivalence is not the purpose of the load segmentation. The purpose of the load segmentation and the section segmentation is to increase the strain energy stored in the system. This is why, there is no requirement about equivalence of moments. The only requirement is the sum of the magnitudes of the segmented forces P_i to be equal to the magnitude of the original loading force *P* and the sum of the cross-sectional areas A_i of the segmented bars to be equal to the cross-sectional area *A* of the original bar.

Because, the supports in Figure 2*a*,*b* are rigid, no deformations exist in the support and the strain energy of the system of bars is a sum of the strain energies $P_i / (2k_i)$ of the individual bars only. As a result, the total strain energy U_{seg} of all segmented bars in Figure 2*b* is: $U_{seg} = P_1^2 / (2k_1) + ... + P_n^2 / (2k_n)$ while the strain energy U_0 of the solid bar in Figure 2*a* is: $U_0 = P / [2(k_1 + k_2 + ... + k_n)]$. Calculating the total strain energy of the system of individual bars in Figure 2b has been done by applying Equation (18) *n* times, where *n* is the number of segmented bars. The same equation (18) for the strain energy is valid for both a solid bar with a particular cross section and for every single individual bar with a smaller cross section.

The two strain energies can now be compared through inequality (11). If $b_i = k_i$, inequality (11) can be rewritten as

$$\frac{P^2}{2(k_1 + k_2 + ... + k_n)} \leq \frac{P_1^2}{2k_1} + ... + \frac{P_n^2}{2k_n} \tag{19}$$

Inequality (19) can now be interpreted meaningfully in the next result:

The accumulated strain energy due to a force P acting on a single bar is smaller than the accumulated strain energy, resulting from segmenting the original force P into smaller forces and applying the smaller forces P_i to the individual bars into which the original bar has been segmented.

It needs to be pointed out that not every load segmentation leads to an increase of the accumulated strain energy. If all loaded bars have the same displacement, no increase in the accumulated strain energy is present. Indeed, in inequality (19), the value $\delta_i = P_i / k_i$ is the displacement of the *i*th bar under load P_i. If all bars displacements are equal ($\delta_1 = \delta_2 = ... = \delta_n = \delta$), it is easy to verify that equality is attained in (19). Indeed, $\frac{P_1^2}{k_1} + ... + \frac{P_n^2}{k_n} = \delta(P_1 + P_2 + ... + P_n) = \delta \times P$.

Since, $P/(k_1 + ... + k_n) = \delta$, and $P^2/(k_1 + ... + k_n) = \delta \times P$, equality is indeed attained in (19). Note that the loading in Figure 2a and 2b are *completely different*. In Figure 2b, the bar and the force have been split into smaller bars and forces and each individual bar has been loaded independently. As a result, the stresses and displacement across the segmented bars are no longer uniform.

Inequality (19) holds irrespective of the magnitude of the separate forces into which the initial force *P* has been segmented and the cross sections and material properties of the individual bars. The inequality provides a method of increasing the capacity for absorbing strain energy upon dynamic loading.

This is unexpected result. To illustrate its validity, consider an example of a steel bar with cross-section 10mm x 20mm and length 1m, loaded at one end with a common force of 50 kN. The Young modulus of the steel is 210 GPa and the yield strength of the material is 650 MPa.

According to equation (18), the accumulated strain energy in the bar with stiffness $k = EA/L$, loaded with a common force $P_0 = 50$ kN is given by

$$U_0 = \frac{P_0^2}{2k} = \frac{P_0^2}{2k} = \frac{(50 \times 10^3)^2}{2 \times [(210 \times 10^9 \times 20 \times 10 \times 10^{-6})/1]} = 29.76J \qquad (20)$$

Now, suppose that the bar has been segmented into two bars with cross-sections 10mm x 10mm and length 1m, and the force P_0 has been segmented into two *unequal* forces with magnitudes 40 kN, and 10 kN, applied to the individual bars. In this case, the accumulated strain energy in the bars is

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{(40 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6}/1]} + \frac{(10 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6}/1]} \qquad (21)$$
$$= 38.1 + 2.38 = 40.48J$$

This is about 40% larger than the strain energy of 29.76J accumulated in the single bar.

Increasing the capability of accumulating strain energy is important not only in cases of preventing failure during dynamic loading but also in cases where strain energy needs to be stored in elastic elements working in parallel, for a subsequent release.

It needs to be pointed out that not every type of load and bar segmentation leads to an increase in the accumulated strain energy.

If the bar has been segmented into two bars with cross sections 8mm x 10mm and 12mm x10mm and the force has been segmented into two forces with magnitudes 20kN and 30kN, the accumulated strain energy characterising the segmented bar and loading forces is equal to the strain energy characterising the original bar and loading force:

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{(20 \times 10^3)^2}{2 \times [210 \times 10^9 \times 8 \times 10 \times 10^{-6}/1]} + \frac{(30 \times 10^3)^2}{2 \times [210 \times 10^9 \times 12 \times 10 \times 10^{-6}/1]} = 29.76J$$

(22)

This is because the displacements of the segmented bars are equal (the ratios $P_1/k_1 = P_2/k_2 = \delta$, are the same) and, according to the basic properties of inequality (19), in this case, equality is attained.

If the bar has been segmented into two bars with cross sections 10mm x 10mm and the force has been segmented into two equal forces with magnitude 25 kN, again, the accumulated strain energy in the bar will be the same:

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = 2 \times \frac{(25 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6} / 1]} = 29.76J \tag{23}$$

because the displacements of the bars $P_1 / k_1 = P_2 / k_2 = \delta$ are the same.

In order to obtain advantage from inequality (19), *asymmetry must be present in the syste*m so *that the displacements of the bars are not equal*: $\delta_1 = P_1 / k_1 \neq \delta_2 = P_2 / k_2$. Existence of asymmetry is essential for increasing system's performance through segmentation based on inequality (19). Segmented bars where the individual bars experience the same displacement do not yield an increase of the amount of stored strain energy.

Consider the sequence $\{P_1^2, P_2^2, ..., P_n^2\}$ and the sequence $\{1/(2k_1), 1/(2k_2), ..., 1/(2k_n)\}$. The dot product of these sequences is the right-hand side of inequality (19). According to the rearrangement inequality, the dot product of two sequences is maximum if they are similarly ordered, for example if $P_1^2 \leq P_2^2 \leq ... \leq P_n^2$ and $1/(2k_1) \leq 1/(2k_2) \leq, ..., \leq 1/(2k_n)$. For the first sequence, it can be shown that if $P_1^2 \leq P_2^2 \leq ... \leq P_n^2$ then $P_1 \leq P_2 \leq ... \leq P_n$. Indeed, from the basic properties of inequalities, for positive P_i and P_j from $P_i^2 \leq P_j^2$ it follows that $P_i \leq P_j$. For the second sequence, it is easy to see that $1/k_1 \leq 1/k_2 \leq, ..., \leq 1/k_n$ only if $k_1 \geq k_2 \geq ... \geq k_n$. For any other permutation of the sequences, for example $\{P_{k1}, P_{k2}, ..., P_{kn}\}$ and $\{k_{s1}, k_{s2}, ..., k_{sn}\}$, the next inequality is fulfilled:

$$\frac{P_1^2}{2k_1} + ... + \frac{P_n^2}{2k_n} \geq \frac{P_{k1}^2}{2k_{s1}} + ... + \frac{P_{kn}^2}{2k_{sn}} \tag{24}$$

As a result, the right-hand side of inequality (19) is maximised if the ordered in ascending order segments $P_1 \leq P_2 \leq ... \leq P_n$ of the loading force are matched with the ordered in descending order stiffness values: $k_1 \geq k_2 \geq ... \geq k_n$.

The meaningful interpretation of the abstract inequality (11) helped to find another overlooked property in the mature field of mechanical engineering. No such result has been reported in modern comprehensive textbooks in the area of mechanical engineering and stress analysis (Collins, 2003; Norton, 2006; Pahl, 2007; Childs, 2014; Budynas, 1999,2015; Mott et al, 2018; Gullo and Dixon, 2018; Gere and Timoshenko, 1999).

5. Meaningful interpretation of single-variable sub-additive and super-additive functions

The use of single-variable sub-additive and super-additive inequalities in process optimisation will be illustrated by power-type dependences involving a single controlled factor. Power functions are widespread (Andriani and McKelvey, 2007; Newman, 2007; Easley and Kleinberg, 2010). Many extensive quantities are approximated very well by power functions of the type

$$y = ax^p \tag{25}$$

where *a*, *p* are constants ($a \neq 0; p > 0$), *x* is the controlling factor ($x \geq x_0$) and *y* is the output quantity.

The power functions given by equation (25) are encountered frequently in modelling various phenomena, where *y* stands for frequency, energy, power, force, damage, profit, pollution, etc. Power laws appear, for example, in cases where positive feedback loops are determining the output. At the heart of these positive feedback loops is usually the *preferential attachment*, according to which a commodity is distributed according to how much commodity is already present.

Depending on whether the power *p* in (25) is greater or smaller than unity, sub-additive or super-additive inequalities can be used to perform segmentation or aggregation of extensive controlling factor in order to attain enhanced performance.

The output quantity *y* can be a convex function of the controlling factor *x* ($x \geq 0$) depending on whether the second derivative $d^2y/dx^2 = ap(p-1)x^{p-2}$ with respect to *x* is positive or negative. This depends on whether the power *p* is greater or smaller than 1. If $p > 1$, $d^2y/dx^2 \geq 0$, and the power function *y* is convex. If $0 < p < 1$, $d^2y/dx^2 \leq 0$ and the power function *y* is concave. If $p < 0$, $d^2y/dx^2 = ap(p-1)x^{p-2} > 0$ and the power function *y* is convex.

Suppose that the output function *y* is of the type presented by equation (25). For a controlling factor *x* varying in the interval $[0,\infty)$ and output quantity *y* varying within the range $[0,\infty)$, the function (25) is convex if $p > 1$ and since $y(0) = 0$, the following super-additive inequality holds:

$$a(x_1 + x_2 + ... + x_n)^p \geq ax_1^p + ax_2^p + ... + ax_n^p \qquad (26)$$

This inequality means that aggregating the extensive factor *x* into *n* parts yields a larger total effect.

If $0 < p < 1$ the power function (25) is concave and the following sub-additive inequality holds:

$$a(x_1 + x_2 + ... + x_n)^p \leq ax_1^p + ax_2^p + ... + ax_n^p \qquad (27)$$

which means that segmenting the controlling factor *x* yields a larger total effect.

Consider now an application example involving again elastic strain energy accumulated by elastic elements. Suppose that the elastic strain energy accumulated due to a displacement *x* is given by the power function:

$$U = a\,x^p \qquad (28)$$

where *a* and *p* ($p > 1$) are constants that depend on the material and its elastic properties. The extensive factor in the power-law dependence (28) is the displacement *x* of the loaded elastic element.

For $p > 1$, the extensive quantity *U* of accumulated elastic strain energy is a convex function of the displacement because the second derivative with respect to the displacement *x* is positive: $d^2U/dx^2 = ap(p-1)x^{p-2} > 0$. Because for a displacement varying in the interval $[0,\infty)$, the elastic strain energy is within the range $[0,\infty)$ and because $U(0) = 0$, the function (28) giving the elastic strain energy is super-additive. Consequently, the inequality

$$a(x_1 + x_2 + ... + x_n)^p \geq ax_1^p + ax_2^p + ... + ax_n^p \qquad (29)$$

holds. The right side of inequality (29) can be interpreted as the sum of the elastic strain energy accumulated in *n* elastic elements subjected to displacements $x_1, x_2, ..., x_n$ while the left-hand side can be interpreted as quantity of potential energy accumulated in a single elastic element subjected to displacement equal to the sum of the displacements $x_1, x_2, ..., x_n$ of the separate elastic elements.

As a result, the interpretation of inequality (29) leads to the following useful result: The elastic energy stored in *n* elastic elements with displacements $x_1, x_2, ..., x_n$, respectively, is smaller than the elastic energy stored in a single elastic element with displacement of magnitude *x*, equal to the sum $x = x_1 + x_2 + ... + x_n$ of the displacements of all multiple elastic elements.

As a simple illustration the super-additive inequality (29), consider a non-linear, fully elastic element where the loading force *F* varies with the displacement δ, according to the dependence:

$$F(\delta) = a\sqrt{\delta} \tag{30}$$

where $a > 0$ is a material constant. The accumulated elastic energy if the displacement varies within the limits $0 \le \delta \le x$ is given by

$$U(x) = \int_0^x F(\delta) d\delta = \int_0^x a\sqrt{\delta}\, d\delta = (2/3)ax^{3/2} \tag{31}$$

Because $p > 1$ in (28), inequality (29) becomes:

$$(2/3)a(x_1 + x_2 + ... + x_n)^{3/2} / 2 \ge (2/3)ax_1^{3/2} + (2/3)ax_2^{3/2} + ... + (2/3)ax_n^{3/2} \tag{32}$$

Consider now an application example from economics involving a sub-additive function involving extensive quantities. Suppose that the annual profit *z* from an investment in a particular enterprise is given by the power function:

$$z = c\,x^q \tag{33}$$

where *x* is the size of the investment, *c* and *q* ($q < 1$) are constants which depend on the particular enterprise. The extensive factor in the power-law dependence (33) is the size of the investment *x*. The profit z is also extensive quantity.

For $0 < q < 1$, the profit *z* is a concave function of the size of investment because the second derivative with respect to the displacement *x* is negative: $d^2z / dx^2 = cq(q-1)x^{q-2}$. Because for investment varying in the interval $[0, \infty)$, the profit varies in the range $[0, \infty)$ and $z(0) = 0$, function (33) giving the profit is sub-additive. Consequently, inequality (27) becomes

$$c(x_1 + x_2 + ... + x_n)^q \le ax_1^q + ax_2^q + ... + ax_n^q \tag{34}$$

The interpretation of inequality (34) leads to an important conclusion: splitting the initial investment *x* and investing in *n* parallel enterprises yields larger profit than investing the entire sum *x* in a single enterprise. The difference in profit can be significant as can be verified from the next numerical example. For a profit dependence

$$z = 15.3x^{0.42} \tag{35}$$

the profit from investing $x = \$10000$ is \$732.3. Splitting the investment in two and investing \$5000 in two parallel enterprises gives $z = 2 \times (15.3 \times 5000^{0.42}) = \$1094.7$ which is by 50% larger than the profit attained from the initial investment.

6. Generating new knowledge by interpreting algebraic inequalities in terms of potential energy

Consider the general abstract inequality

$$f(x_1, x_2, ..., x_n) \ge L \tag{36}$$

where $x_1, x_2, ..., x_n$ are system/process parameters and *L* is an unknown lower bound. The parameters may be subjected to a constraint:

$$\varphi(x_1, x_2, ..., x_n) = 0$$

where $\varphi(x_1, x_2, ..., x_n) = 0$ is an arbitrary continuous function of the system/process parameters $x_1, x_2, ..., x_n$. A common constraint is $\varphi \equiv \sum_{i=1}^{n} x_i = d$, where *d* is a constant.

Often, the function $f(x_1, x_2, ..., x_n)$ can be interpreted as *potential energy* of the system. If this can be done, the constant *L* in the right-hand of inequality (36) effectively represents the system state of stable equilibrium which corresponds to a minimum potential energy. From the forces/moments or other conditions that correspond to the stable equilibrium, a number of useful relationships can be derived and the lower bound *L* can be determined without resorting to complex models.

In interpreting the left hand side of inequality (36) as potential energy, physical analogies such as *constant tension springs*, *zero-length linear springs* and *zero-length non-linear springs* prove to be useful.

A *constant-tension* spring is a spring whose tension is independent of its length (Levi, 2009). It is assumed that the tension *T* in the constant-tension spring when the system is in equilibrium, is equal to unity (*T=1*).

The potential energy *E* of a constant-tension spring of length *x* is given by

$$E = \int_0^x T du = Tx = x \tag{37}$$

which is product of the spring length *x* and the tension of the spring *T* (*T*=1).
Note that the potential energy of a constant-tension string ($E = x$) as a function of the spring length *x* is different from the potential energy $E = (1/2)x^2$ of a conventional spring stretched to a displacement *x*. This is because for a conventional spring, the spring force $F = kx$ varies proportionally with the displacement *x*, while for a constant-tension spring, the spring force is constant $F = 1$ and does not depend on the displacement *x*. Such a spring can be constructed with weights of unit magnitude and pulleys (Figure 3b). For constant weights, the spring force is always constant. The potential energy however, varies with the spring length. The larger the length $KA + KB$ of the constant-tension spring (Figure 3b), the larger is the sum of the elevations of the suspended unit weights and the larger is the potential energy of the system.

To illustrate the idea behind the potential energy as a basis for meaningful interpretation of algebraic inequalities, a simple example will be used. Consider the inequality:

$$\sqrt{a^2 + x^2} + \sqrt{b^2 + y^2} \geq L \tag{38}$$

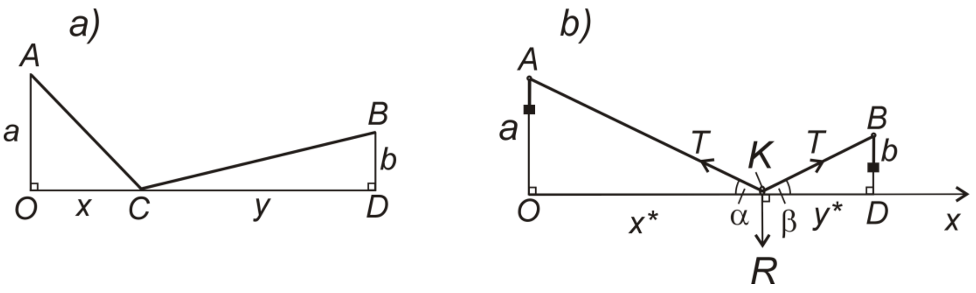
where the non-negative parameters *x,y* are subjected to the constraint

$$x + y = d$$

where $d = OD$ (Figure 3a) and *a,b* and are given constants. In inequality (38), the lower bound *L* is unknown quantity.

Noticing that both $\sqrt{a^2 + x^2}$ and $\sqrt{a^2 + y^2}$ can be interpreted as a hypotenuse of a right-angle triangle, inequality (38) can be presented by using the two right-angle triangles OAC and CBD in Figure 3a:

Figure 3. Meaningful interpretation of inequality (38) in terms of potential energy of a constant-tension spring.



The left hand side of inequality (38) is the length of ACB and the lower bound L is the smallest value of the length ACB.

Suppose that K in Figure 3b is a ring that moves without any friction along the segment OD, whose length is equal to d and AKB is a non-stretchable string kept under constant tension by the pulleys A and B and the suspended unit weights at the ends. The left part of inequality (38) is then proportional to the potential energy of the system in Figure 3b. The smallest potential energy of the system in Figure 3b is associated with the smallest length $KA+KB$ of the string, which corresponds to the smallest combined elevation of the unit weights.

The constant L on the right part of inequality (38) represents effectively the minimum potential energy of the system. For a state of stable equilibrium, the forces applied to point K on the string must balance out. Because the ring is frictionless, the force R applied to the string from the ring is perpendicular to the x-axis and its component along the x-axis is zero. Consequently, the sum of the components of the tension force T applied on the string at point K, along the x-axis, must be equal to zero ($-T\cos\alpha + T\cos\beta = 0$). From this necessary condition for a system equilibrium, it is clear that the angle AKO (α) must be equal to angle BKD (β). In this case, triangles AKO and BKD are similar and $x^*/a = (d-x^*)/b$. From this relationship, $x^* = ad/(a+b)$ and $y^* = bd/(a+b)$. The value of the lower bound L in inequality (38) is equal to the smallest length $L = (a+b)\sqrt{1+d^2/(a+b)^2}$ of AKB. As a result, the exact value of the lower bound L in inequality (38) has been determined solely on the basis of the meaningful interpretation of inequality (38) in terms of potential energy.

6.1 A necessary condition for minimising sum of the powers of distances by a meaningful interpretation of an inequality

Consider n points in space: A_1, A_2,..., A_n and an extra point M with distances r_1, r_2,..., r_n to the points A_1, A_2,..., A_n, respectively (Figure 4a).

Consider the abstract inequality (39) involving the distances r_1, r_2,..., r_n:

$$r_1^n + r_2^n + \ldots + r_n^n \geq L \tag{39}$$

where L is the lower bound which is unknown quantity.

Suppose also that the connecting segments $r_i = M_0A_i$ (Figure 4b) are *non-linear zero-length tension springs* of order n-1. A nonlinear zero-length spring of order n-1 is a spring whose tension T is directly proportional to the n-1 power of its length r ($T = kr^{n-1}$). The potential energy u of a stretched non-linear zero-length spring of order n-1, to a length r, is given by

$$u = \int_0^r kv^{n-1}dv = kr^n / n \qquad (40)$$

From equation (40), it can be seen that if the constant *k* is set to be equal to *n*, the left part of inequality (39) can be meaningfully interpreted as the total potential energy of a system of *n* non-linear zero-length springs of order *n*-1 while the right part of inequality (39) can be interpreted as the minimum possible potential energy of the system of non-linear zero-length springs of order *n*-1.

If point *M* is now selected in such a way (point M_0) that the sum of the *n*th power of the distances r_1, r_2,..., r_n is a minimum, the equality

$$r_1^n + r_2^n + ... + r_n^n = L \qquad (41)$$

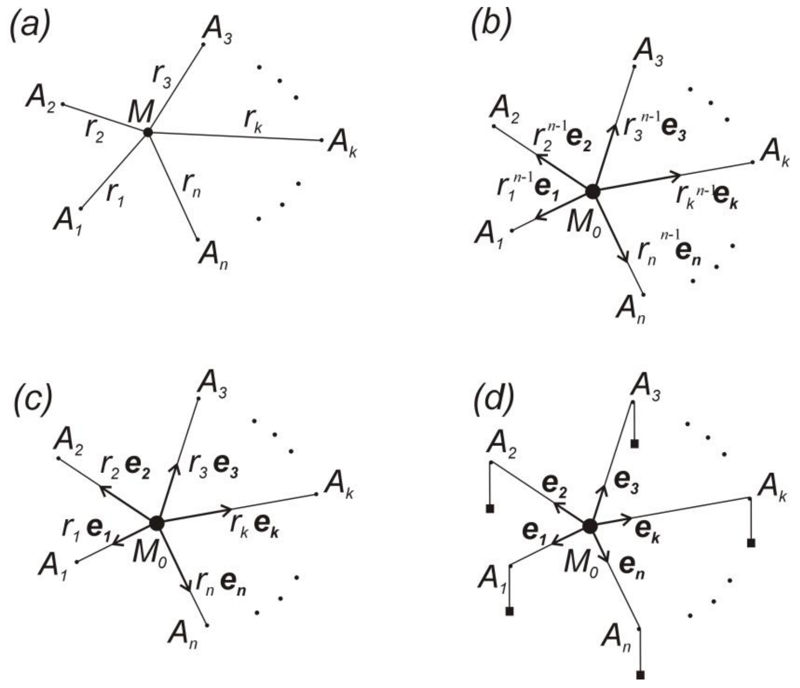
holds.

Since the minimum potential energy corresponds to a stable equilibrium of the system, the sum of the vectors $r_i^{n-1}\mathbf{e_i}$ with origin point M_0, directed towards the separate points A_1, A_2 ,..., A_n, is zero (Figure 4b):

$$\sum_{i=1}^{n} r_i^{n-1}\mathbf{e_i} = 0 \qquad (42)$$

As can be seen from Figure 4b, the meaningful interpretation of the left and right part of inequality (39) led to a necessary condition regarding the optimal location M_0. The point M_0 corresponding to the minimal sum of distances to the specified points A_i raised into power of *n*, coincides with the point for which $\sum_{i=1}^{n} r_i^{n-1}\mathbf{e_i} = 0$. There are two important special cases of inequality (39) which correspond to $n = 2$ and $n = 1$.

Figure 4. (a) Distances MA_i from point *M* to a specified set of points in space (b) The necessary condition for a minimal sum of distances M_0A_i raised to the power of *n* is the sum of the vectors $r_i^{n-1}e_i$ to add up to zero; (c) The necessary condition for a minimal sum of squared distances M_0A_i is the location of point M_0 in the mass center of the system of points; (d) The necessary condition for a minimal sum of distances M_0A_i is the location of point M_0 in the geometric median of the system of points.



6.2 Special case 1: Determining the lower bound of the sum of squared distances to a specified number of points in space

Consider a special case of the inequality (39) where $n = 2$. In this case, inequality (39) transforms into the inequality

$$r_1^2 + r_2^2 + ... + r_n^2 \geq L \qquad (43)$$

In this case, the connecting segments $r_i = M_0 A_i$ can be interpreted as *zero-length linear tension springs*. The tension of a linear zero-length tension spring is directly proportional to its length *r* ($T = kr$). According to equation (40), the potential energy of a stretched zero-length linear spring to a length *r* is given by $(1/2)kr^2$. If the spring stiffness *k* is selected to be equal to 2 ($k = 2$), the left part of inequality (43) expresses the potential energy of a system of *n* zero-length linear springs multiplied by 2 while the right part of inequality (43) corresponds to the minimum potential energy of the system of zero-length linear tension springs. Since the minimum potential energy corresponds to a stable equilibrium of the system, the sum of the vectors $r_i \mathbf{e_i}$ from point M_0 directed towards the separate points A_1, A_2,..., A_n must be zero: $\sum_{i=1}^{n} r_i \mathbf{e}_i = 0$ (see Figure 4c). This means that, in this case, point M_0 must coincide with the mass centre of the system of points.

As can be seen from Figure 4c, the meaningful interpretation of the left and right part of inequality (41) for $n = 2$ led to a necessary condition regarding the location of point M_0. The point with the minimal sum of squared distances to the fixed points coincides with the mass centre $\mathbf{r_c}$ of the system of points, for which the well-known dependence $\sum_{i=1}^{n} (\mathbf{r_c} - r_i \mathbf{e_i}) = 0$ holds. From the last dependence, the coordinates of the mass centre of a system of points are determined by the well-known relationships: $\mathbf{r_c} = (1/n)\sum_{i=1}^{n} r_i \mathbf{e_i}$; $r_{c,x} = (1/n)\sum_{i=1}^{n} r_{i,x}$; $r_{c,y} = (1/n)\sum_{i=1}^{n} r_{i,y}$, $r_{c,z} = (1/n)\sum_{i=1}^{n} r_{i,z}$.

6.3 Special case 2: A necessary condition for determining the lower bound of the sum of distances

Consider now a special case of the inequality (39), where $n=1$. In this case, inequality (39) transforms into the inequality

$$r_1 + r_2 + ... + r_n \geq L \tag{44}$$

Inequality (44) can also be interpreted in a meaningful way. Suppose that the connecting segments $M_0 A_i$ are constant-tension springs (Figure 4d). Assume that the tension *T* in each of these strings, when the system is in equilibrium, is equal to unity (*T=1*).

According to equation (40), the potential energy of a stretched constant-tension spring to a length *r* is given by kr, which is product of the spring length and the tension of the spring ($k=1$). The constant-tension string has been constructed in Figure 3d by the weights of unit magnitude and pulleys. The left part of inequality (44) is then proportional to the potential energy of the system in Figure 4d. Indeed, the larger the sum of distances $M_0 A_i$ to the individual points is, the larger is the sum of the elevations of the suspended unit masses and the larger is the potential energy of the system. A minimum potential energy of the system is attained if and only if the sum of the distances $M_0 A_i$ to the points is minimal.

Since the minimum potential energy corresponds to a stable equilibrium of the system, the sum of the unit vectors e_i from point M_0 directed towards the separate points must be zero: $\sum_{i=1}^{n} e_i = 0$ (see Figure 4d). As a result, the meaningful interpretation of the left-hand and right-hand of inequality (44) leads to a necessary condition regarding the location M_0. The point with minimal sum of the distances to the points A_1, A_2 ,..., A_n is a point for which the sum of the unit *line-of-sight* vectors towards the points is zero: $\sum_{i=1}^{n} e_i = 0$. In this case, the optimal point M_0 coincides with the *geometric median* (Wesolowsky, 1993) of the system of points. Finding the location M_0 that minimises the sum of the distances to the specified points is the famous *Fermat-Weber location problem* (Chandrasecaran and Tamir, 1990). Algorithms for determining the location M_0 have been presented in (Weiszfeld, 1937; Chandrasecaran and Tamir, 1990; Papadimitriu and Yanakakis, 1982).

7 Conclusions and further development

The paper demonstrates that sub-additive and super-additive multivariate functions can be used to generate new knowledge and enhance the impact of extensive quantities in unrelated domains of engineering by segmenting or aggregating extensive quantities.

Segmentation of extensive quantities through multivariate sub-additive and super-additive functions has also been used to optimise systems and the presented general approach based on sub-additive and super-additive functions is applicable to any area of science and technology.

Indeed, in electrical engineering, the meaningful interpretation of a sub-additive function which is effectively the modified Cauchy-Schwarz inequality, led to a method for increasing the power output from a single voltage source, on elements connected in series. The power output on such elements is smaller than the total power output from the segmented source applied to the separate elements.

In mechanical engineering, the meaningful interpretation of the modified Cauchy-Schwarz inequality led to a method for increasing the capacity of absorbing strain energy. Increasing the capacity of absorbing strain energy can be done by segmenting the load and the component and applying the segmented loads to the segmented smaller components.

It needs to be pointed out that the existence of asymmetry is essential to increasing the strain energy absorbing capacity and power output. Loaded elements experiencing the same displacement do not yield an increase of the absorbed strain energy. Similarly, loaded resistances experiencing the same current do not yield an increase of the power output.

The paper also established important properties for processes described by power laws $y = ax^p$. For an exponent greater than unity ($p > 1$), the segmentation of the controlling factor *x* leads to a smaller total yield from the process. For an exponent smaller than unity ($0 \le p < 1$) aggregation of the controlling factor *x* leads to a smaller total yield from the process.

Finally, the meaningful interpretation of an algebraic inequalities in terms of potential energy resulted in a general necessary condition for minimising the sum of powers of distances to a fixed number of points in space.

The developments in this paper can be extended in a number of different ways:

- Developing other application studies from different domains by providing alternative interpretation of the presented inequalities
- Developing other multivariate sub-additive and super-additive inequalities involving extensive quantities and applying them to generate new knowledge and optimising system performance in new application domains.
- Developing new general multivariate inequalities involving both extensive and intensive quantities and applying them to generate new knowledge and optimising system performance in new application domains.
- Extending the potential energy approach for determining the sum of distances to a set planar objects or three-dimensional objects.

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