

# Meaningful interpretation of algebraic inequalities to achieve uncertainty and risk reduction

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## Abstract

The paper develops an important method related to using algebraic inequalities for uncertainty and risk reduction and enhancing systems performance. The method consists of creating relevant meaning for the variables and different parts of the inequalities and linking them with real physical systems or processes. The paper shows that inequalities based on multivariable sub-additive functions can be interpreted meaningfully and the generated new knowledge used for optimising systems and processes in diverse areas of science and technology. In this respect, an interpretation of the Bergström inequality, which is based on a sub-additive function, has been used to increase the accumulated strain energy in components loaded in tension and bending. The paper also presents an interpretation of the Chebyshev's sum inequality that can be used to avoid the risk of overestimation of returns from investments and an interpretation of a new algebraic inequality that can be used to construct the most reliable series-parallel system. The meaningful interpretation of other algebraic inequalities yielded a highly counter-intuitive result related to assigning devices of different types to missions composed of identical tasks. In the case where the probabilities of a successful accomplishment of a task, characterising the devices, are unknown, the best strategy for a successful accomplishment of the mission consists of selecting randomly an arrangement including devices of the same type. This strategy is always correct, irrespective of existing unknown interdependencies among the probabilities of successful accomplishment of the tasks characterising the devices.

## Keywords

Uncertainty reduction, risk reduction, algebraic inequalities, system reliability, decisions under uncertainty

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## Introduction

Algebraic inequalities, have been used extensively in mathematics. For a long time, simple inequalities have been used to express error bounds in approximations and constraints in linear programming models. The properties of a number of useful non-trivial algebraic inequalities, such as the Arithmetic mean – Geometric mean (AM-GM) inequality, Cauchy-Schwarz inequality, the rearrangement inequality, the Chebyshev's inequalities, Jensen's inequality, Muirhead's inequality, Hoelder's inequality, etc., have also been well documented.<sup>1–8</sup>

In reliability and risk research, inequalities have been used exclusively as a tool for reliability and risk evaluation and for characterisation of reliability functions.<sup>9–15</sup> It is important to guarantee that the reliability of a system meets certain minimal expectations and inequalities have been used<sup>9</sup> for obtaining lower and upper bounds on the system reliability by using

minimal cut sets and minimal path sets. Xie and Lai,<sup>10</sup> for example, used simple conditional inequalities to obtain more accurate approximations for system reliability, instead of the usual minimal cut and minimal cut bounds. By using improved Bonferroni inequalities, lower and upper bound of system reliability have been derived.<sup>11</sup> Inequality-based reliability estimates for complex systems have also been proposed.<sup>12</sup>

In the reliability and risk research, algebraic inequalities have also been used to express relationships between random variables and their transformations to generate insight into the structure of reliability

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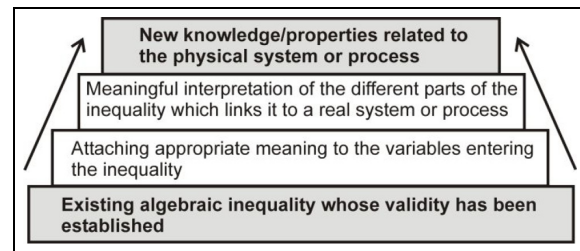
distributions. Simple inequalities with relation to reliability prediction have been used<sup>13</sup>; inequalities involving expectations have been used for characterising some well-known reliability distributions.<sup>14</sup> Well-known inequalities about a random variable with unknown probability distribution are the Chebyshev's inequality and Markov's inequality.<sup>16</sup>

The reliability-related applications of inequalities however, are very limited and oriented towards measuring the reliability performance of the systems, instead of providing direct input to the design process by reducing uncertainty and risk and improving the reliability of components, systems and processes. In standard reliability textbooks<sup>17–20</sup> there has been a big gap in applying algebraic inequalities to improve reliability and reduce risk. Recently, this gap has been reduced<sup>35</sup> by demonstrating the use of algebraic inequalities for reliability improvement and risk reduction.

Applications of inequalities have also been considered in physics<sup>21</sup> and engineering.<sup>22,23</sup> In the mechanical engineering design literature,<sup>23–31</sup> for example, there has been a lack of discussion on the use of algebraic inequalities to enhance systems and process performance. In engineering design, there is also a big gap in the application of algebraic inequalities for systems and process enhancement. The application of algebraic inequalities has been mainly confined to trivial inequalities linking design variables required to satisfy various trivial design constraints in order to guarantee that the design will perform its required functions.<sup>23</sup> Recently, this gap in using algebraic inequalities has been reduced<sup>36</sup> by demonstrating the use of algebraic inequalities for improving the performance of processes described by a power-law dependence. It was shown that the application of inequalities in engineering is far reaching and is not restricted to specifying design constraints.

A formidable advantage of algebraic inequalities is that they present an effective tool for dealing with deep, unstructured uncertainty related to key reliability-critical parameters of systems and processes. Algebraic inequalities *do not require knowledge related to the distributions of the variables entering the inequalities* and this makes the method based on algebraic inequalities ideal for handling deep uncertainty.<sup>35</sup> In this respect, algebraic inequalities avoid a major difficulty in the conventional models for handling uncertainty – lack of meaningful specification of frequentist probabilities or lack of justification behind the assigned subjective probabilities and probabilistic models.

Thus, by using inequalities, the reliabilities of two systems can be compared in the absence of knowledge about the reliabilities of the separate components or in the presence of partial knowledge only, for example, that a particular type of component is older (less reliable) than another type of component.<sup>35</sup> It needs to be pointed out that algebraic inequalities not only permit ranking two systems in terms of reliability but also identifying the optimal (most reliable) system.<sup>34,35</sup>



**Figure 1.** Deriving new knowledge/properties by meaningful interpretation of a valid algebraic inequality.

Algebraic inequalities provide a strong support for risk-critical decisions under deep, unstructured uncertainty. Despite that the distributions of the risk-critical parameters remain unknown, the method of algebraic inequalities can still establish the intrinsic superiority of one of the competing options.

Algebraic inequalities can also be used for determining tight upper or lower bounds for the variation of risk-critical parameters. Complying the design with these bounds helps maximise performance and avoid faulty products and failures.

These advantages of algebraic inequalities have already been demonstrated.<sup>32–35</sup>

Unlike algebraic equalities, which establish that two different configurations of a system or a process are equivalent, an algebraic inequality establishes that one of the compared configurations is superior. In this way, the use of algebraic inequalities opens wide opportunities for enhancing the performance of systems and processes.

The applications of algebraic inequalities discussed earlier<sup>32–36</sup> are based on the forward approach which includes several basic steps: (i) detailed analysis of the system (e.g. by using reliability theory), (ii) conjecturing an inequality about the competing alternatives or an inequality related to the bounds of a risk-critical parameter, (iii) testing the conjectured inequality by using Monte Carlo simulation and (iv) proving the conjectured inequality rigorously. This way of exploiting algebraic inequalities has already been demonstrated,<sup>34,35</sup> with comparing systems with unknown reliabilities of their components.

Another formidable advantage of algebraic inequalities is that they also admit meaningful interpretation that can be attached to a real system or process. Furthermore, depending on the specific interpretation, knowledge applicable to different systems from different domains can be released from the same inequality. This is the essence of inverse approach<sup>37</sup> in applying algebraic inequalities, which starts with a correct algebraic inequality, continues with creating relevant meaning for the variables and the different parts of the inequality and ends with formulating undiscovered properties/knowledge about the system or process. This approach has been illustrated in Figure 1.

The inverse approach effectively links existing correct abstract algebraic inequalities with real physical

systems or processes and not only opens opportunities for enhanced performance of systems and processes but also leads to the discovery of new fundamental properties.

The inverse approach related to creating meaningful interpretation for existing non-trivial abstract inequalities and attaching it to a real system or process has only been partially explored in Todinov.<sup>36,37</sup> Thus, an interpretation of a multi-variable algebraic inequality with respect to real systems has been conducted in Todinov<sup>37</sup> but it was confined to simple electrical circuits and systems of capacitors arranged in series and parallel.

This paper builds on the research on the inverse approach in using algebraic inequalities initiated in Todinov<sup>36,37</sup> and contributes a number of new applications of non-trivial multi-variable algebraic inequalities for reducing risk and enhancing the performance of mechanical systems. These contributions are as follows: (i) improving the reliability of a parallel-series system by interpreting a new non-trivial algebraic inequality, (ii) maximising the probability of a successful accomplishment of missions composed of identical tasks by interpreting non-trivial algebraic inequalities; (iii) increasing the accumulated strain energy in components subjected to tension and bending by interpreting the Bergström's inequality and (iv) avoiding the risk of overestimating profits by using the Chebyshev's sum inequality.

**The principle of non-contradiction**

The inverse approach related to algebraic inequalities is routed in the principle of non-contradiction: *if the variables and the different parts of a correct abstract inequality can be interpreted as a particular system or process, in the real world, the system or process exhibit properties that are consistent with the abstract inequality.* In other words, the realisation of the process/experiment yields results that do not contradict the algebraic inequality.

The deep connection between physical reality and algebraic inequalities can be illustrated with the next simple example.

Consider the common abstract inequality

$$a^2 + b^2 \geq 2ab \tag{1}$$

which is true for any real numbers  $a, b$  because this inequality can be obtained from the obvious inequality  $(a - b)^2 \geq 0$ . Adding  $2ab$  to both sides of inequality (1) will not change its sign and the result is the inequality

$$(a + b)^2 \geq 4ab \tag{2}$$

Consider the positive quantities  $a, b$  ( $a \geq 0, b \geq 0$ ). Dividing both sides of inequality (2) by the positive value  $(a + b)$  does not alter the direction of inequality (2).

$$a + b \geq 4 \frac{ab}{a + b} \tag{3}$$

Dividing the numerator and denominator of the right hand side of (3) by  $ab$  gives

$$a + b \geq 4 \frac{1}{1/b + 1/a} \tag{4}$$

Now, inequality (4) can be given meaningful interpretation by noticing that if  $a$  and  $b$  denote the resistances of two elements, the left hand side of inequality (4) is the equivalent resistance of the elements connected in series. The right hand side of inequality (4) is the equivalent resistance of the elements connected in parallel, multiplied by 4.

Inequality (4) effectively formulates a physical property: the equivalent resistance of two elements connected in series is at least four times greater than the equivalent resistance of the same elements connected in parallel, irrespective of the resistances of the individual elements.

If physical measurements of the equivalent resistances of series and parallel arrangement are performed, they will only confirm that for any combination of values  $a$  and  $b$  of the resistances of the two elements, the equivalent resistance in series is more than four times greater than the equivalent resistance of the same elements connected in parallel. The physical experiment cannot possibly possess knowledge of algebraic inequality (4) yet all experimental outcomes are in perfect agreement with inequality (4). Why does such an agreement exist? Apparently, the mathematics of inequalities is interwoven in the fabric of the real physical world and the course of physical phenomena and processes is inherently consistent with the predictions of correct algebraic inequalities. A meaningful interpretation of the variables and various parts of a correct algebraic inequality which does not lead to a relevant physical manifestation is therefore not possible. This is the essence of the principle of non-contradiction which underlies deriving new knowledge from the meaningful interpretation of algebraic inequalities.

The principle of non-contradiction not only infers the existence of particular properties but also forbids the existence of other properties that are not consistent with the inequalities.

**Increasing the capacity for absorbing elastic energy during dynamic loading on the basis of a meaningful interpretation of the Bergström inequality**

*Inequalities based on sub-additive and super-additive functions*

Multivariate sub-additive functions have been discussed in Rosenbaum.<sup>38</sup>

In the Euclidean space of one dimension, the sub-additive function  $f(x)$  satisfies the inequality

$$f(x_1 + x_2) \leq f(x_1) + f(x_2) \quad (5)$$

for any pair of points  $x_1$  and  $x_2$  in the domain of definition.

If the direction of the inequality is reversed, the function  $f(x)$  is super-additive:

$$f(x_1 + x_2) \geq f(x_1) + f(x_2) \quad (6)$$

In the Euclidean space of two dimensions, the multi-variate sub-additive function  $f(x, y)$  satisfies the inequality

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2) \quad (7)$$

for any pair of points  $(x_1, y_1)$  and  $(x_2, y_2)$  in the domain of definition.

If the direction of the inequality is reversed, the function  $f(x, y)$  is super-additive:

$$f(x_1 + x_2, y_1 + y_2) \geq f(x_1, y_1) + f(x_2, y_2) \quad (8)$$

Consider now  $n$  pair of points  $(a_i, b_i)$  with positive coordinates  $a_i, b_i$  from the definition domain of the function  $f(\cdot)$ . From definition (7), it follows that

$$f(a_1 + a_2 + \dots + a_n, b_1 + b_2 + \dots + b_n) \leq f(a_1, b_1) + f(a_2, b_2) + \dots + f(a_n, b_n) \quad (9)$$

while from definition (8) it follows that

$$f(a_1 + a_2 + \dots + a_n, b_1 + b_2 + \dots + b_n) \geq f(a_1, b_1) + f(a_2, b_2) + \dots + f(a_n, b_n) \quad (10)$$

Relationships (9) and (10) can be obtained by mathematical induction and, for the sake of brevity, details have been omitted. Inequalities (9) and (10) do not change direction upon any permutation of  $a_i$  and  $b_i$ .

In order to apply inequalities (9) or (10),  $a_i, b_i$  and the quantities  $f(a_i, b_i)$  must all be additive quantities. Inequality (9) states that the effect of the additive quantities  $a = \sum_{i=1}^n a_i$  and  $b = \sum_{i=1}^n b_i$  can be increased by segmenting them into smaller parts  $a_i$  and  $b_i, i = 1, \dots, n$  and accumulating their individual effects  $f(a_i, b_i)$ , represented by the terms on the right-hand side of inequality (9). Similarly, inequality (10) states that the effect of the additive quantities  $a = \sum_{i=1}^n a_i$  and  $b = \sum_{i=1}^n b_i$  can be decreased by segmenting them into smaller parts  $a_i$  and  $b_i, i = 1, \dots, n$  and accumulating their individual effects

$f(a_i, b_i)$ , represented by the terms on the right-hand side of inequality (10). Inequalities (9) and (10) have a universal application in science and technology as long as  $a_i, b_i$  and the terms  $f(a_i, b_i)$  are additive quantities and have meaningful interpretation.

Inequalities similar to (9) and (10) can be derived for any number of factors by using the definition of sub-additive/super-additive functions.

### Increasing the accumulated strain energy for components loaded in tension

A special case of the general sub-additive function (9) is the Bergström inequality

$$\frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n} \leq \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \quad (11)$$

valid for any sequences  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  of positive real numbers. The role of the sub-additive function  $f(a, b)$  in inequality (9) is played by the function  $f(a, b) \equiv a^2/b$  in inequality (11).

Bergström inequality<sup>7,8</sup> is effectively a transformation of the well-known Cauchy-Schwarz inequality

$$(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \dots + x_n^2)(y_1^2 + y_2^2 + \dots + y_n^2) \quad (12)$$

valid for any two sequences of real numbers  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$ .

In inequality (12), equality is present if and only if for any  $i \neq j, x_i y_j = x_j y_i$  are fulfilled.

If the substitutions  $x_i = \frac{a_i}{\sqrt{b_i}}$  ( $i = 1, \dots, n$ ) and  $y_i = \sqrt{b_i}$  ( $i = 1, \dots, n$ ) are made in the Cauchy-Schwarz inequality (12), the result is the inequality

$$\left( \frac{a_1}{\sqrt{b_1}} \sqrt{b_1} + \frac{a_2}{\sqrt{b_2}} \sqrt{b_2} + \dots + \frac{a_n}{\sqrt{b_n}} \sqrt{b_n} \right)^2 \leq ((a_1/\sqrt{b_1})^2 + \dots + (a_n/\sqrt{b_n})^2)((\sqrt{b_1})^2 + \dots + (\sqrt{b_n})^2)$$

which leads to inequality (11). Equality in (11) is attained only if  $a_1/b_1 = a_2/b_2 = \dots = a_n/b_n$ .

Inequality (11) is a special case of the general class of inequalities based on sub-additive functions which have a number of interesting potential applications related to increasing the effect of additive quantities (factors). In order to apply inequality (11),  $a_i, b_i$  and the ratios  $a_i^2/b_i$  must all be additive quantities. Inequality (11) effectively states that the effect of the additive quantities  $a = \sum_{i=1}^n a_i$  and  $b = \sum_{i=1}^n b_i$  can be increased by segmenting them into smaller parts  $a_i$  and  $b_i, i = 1, \dots, n$  and accumulating their individual effects  $a_i^2/b_i$  (represented by the terms on the right-hand side of inequality (11)).

Inequality (11) has a universal application in science and technology as long as  $a_i, b_i$  and the terms  $a_i^2/b_i$  are additive quantities and have meaningful interpretation. The condition for applying inequality (11) is to be possible to present an additive quantity  $p_i$  as a ratio of a square of an additive quantity  $a_i$  and another additive quantity  $b_i$ :  $p_i = a_i^2/b_i$ .

As an example, relevant meaning can be created for the variables  $a_i, b_i$  in inequality (11), if variable  $a_i$ , for example, stands for the additive quantity ‘force’ and variable  $b_i$  stands for the additive quantity ‘area’.

It is a well-known result from mechanics of materials<sup>31,39</sup> that the accumulated elastic strain energy  $U$  of a linearly elastic bar with length  $L$  and cross-sectional area  $A$  is given by the equation:

$$U = \frac{P^2 L}{2EA} \tag{13}$$

where  $E$  [Pa] denotes the Young’s modulus of the material and  $P$  [N] is the magnitude of the loading force. The elastic strain energy  $U$  [J] is an additive quantity.

Equation (13) can be written as

$$U = \frac{P^2}{2k} \tag{14}$$

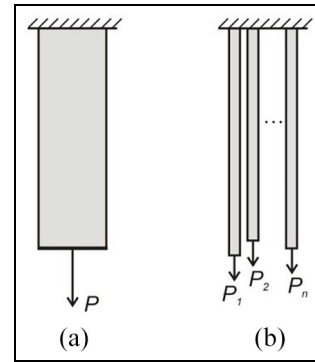
where  $k = EA/L$  [N/m] is the stiffness of the bar.

Consider the two system configurations in Figure 2. In the system configuration in Figure 2(a), a single force  $P$  acts on a single large bar with cross-sectional area  $A$ . In the system configuration in Figure 2(b), the original bar has been segmented into  $n$  individual bars with smaller cross sections  $A_1, A_2, \dots, A_n$ , the sum of which is equal to the cross section  $A$  of the initial bar ( $A = A_1 + \dots + A_n$ ) and the force  $P$  has also been segmented into  $n$  forces  $P_1, P_2, \dots, P_n$ , the sum of which is equal to the initial loading force  $P$  ( $P = P_1 + P_2 + \dots + P_n$ ). The smaller forces  $P_i$  have been applied to the individual bars independently, and the individual bars do not necessarily have the same displacements and stresses (Figure 2(b)). The support is rigid and does not deform due to the segmented forces.

A question of interest is which system configuration in Figure 2 is capable of accumulating more elastic strain energy.

Let  $a_i = P_i, i = 1, \dots, n$  be the forces applied to the centroids of the  $n$  separate bars whose stiffness values are  $k_i = E_i A_i / L_i$  (Figure 2(b)). Let  $b_i = k_i, i = 1, \dots, n$ .

The systems in Figure 2(a) and (b) are not equivalent mechanically and the equivalence is not the purpose of the load segmentation. The purpose of the load segmentation and the section segmentation is to increase the strain energy stored in the system. This is why, there is no requirement about equivalence of moments. The only requirement is the sum of the magnitudes of the segmented forces  $P_i$  to be equal to the magnitude of the original loading force  $P$  and the



**Figure 2.** (a) A force  $P$  applied to a single bar and (b) segmented forces  $P_i$ , applied to the individual bars into which the original bar has been segmented (the individual bars are loaded independently and they do not necessarily have the same displacements and stresses).

sum of the cross-sectional areas  $A_i$  of the segmented bars to be equal to the cross-sectional area  $A$  of the original bar.

Because the supports in Figure 2(a) and (b) are rigid, no deformations exist in the support and the elastic strain energy of the system of bars is a sum of the elastic strain energies  $P_i^2/(2k_i)$  of the individual bars. As a result, the total elastic strain energy  $U_{seg}$  of the multiple bars in Figure 2(b) is  $U_{seg} = P_1^2/(2k_1) + \dots + P_n^2/(2k_n)$  while the elastic strain energy  $U_0$  of the single solid bar in Figure 2(a) is:  $U_0 = P^2/[2(k_1 + k_2 + \dots + k_n)]$ . Calculating the total elastic strain energy of the system configuration of multiple bars in Figure 2(b) has been done by applying equation (14)  $n$  times, where  $n$  is the number of bars. The same equation (14) for the elastic strain energy holds for every single bar with a smaller cross section.

The elastic strain energies of the two system configurations can now be compared through inequality (11). Inequality (11) can be rewritten as

$$\frac{P^2}{2(k_1 + k_2 + \dots + k_n)} \leq \frac{P_1^2}{2k_1} + \dots + \frac{P_n^2}{2k_n} \tag{15}$$

The left- and the right-hand side of inequality (15) can now be interpreted meaningfully in the next result:

*The accumulated elastic strain energy from a force loading a single bar in tension, is smaller than the accumulated strain elastic energy resulting from segmenting the force into smaller forces loading the bars into which the original bar has been segmented.*

It needs to be pointed out that not every load segmentation of the original bar leads to an increase of the accumulated elastic strain energy. If all loaded bars have the same displacement, no increase in the accumulated strain energy is present. Indeed, in inequality (15),  $\delta_i = P_i/k_i$  is the displacement of the  $i$ th bar under load  $P_i$ . If all bars displacements are equal

( $\delta_1 = \delta_2 = \dots = \delta_n = \delta$ ), it is easy to verify that equality is attained in (15). Indeed, the right-hand side of (15) gives  $\frac{1}{2} \left( \frac{P_1^2}{k_1} + \dots + \frac{P_n^2}{k_n} \right) = (1/2)\delta(P_1 + P_2 + \dots + P_n) = (1/2)\delta \times P$ .

Since,  $P/(k_1 + \dots + k_n) = \delta$ , and for the left-hand side of (15)  $P^2/[2(k_1 + \dots + k_n)] = (1/2)\delta \times P$  holds, equality is indeed attained in (15). Note that the loading in Figure 2(a) and (b) are very different. In Figure 2(b), the bar and the force have been split into smaller bars and forces and each individual bar has been loaded independently. As a result, the stresses and displacements across the segmented bars are no longer uniform.

Inequality (15) holds irrespective of the magnitude of the separate forces into which the initial force  $P$  has been segmented and the cross sections of the individual bars. The inequality provides a method of increasing the capacity for absorbing elastic strain energy upon dynamic loading.

This is unexpected result. To illustrate its validity, consider an example of a steel bar with cross-section  $10 \text{ mm} \times 20 \text{ mm}$  and length  $1 \text{ m}$ , loaded in tension with a force of  $50 \text{ kN}$ . The Young's modulus of the steel is  $210 \text{ GPa}$  and the yield strength of the material is  $650 \text{ MPa}$ .

According to equation (14), the accumulated elastic strain energy in the bar with stiffness  $k = EA/L$ , loaded with a force  $P_0 = 50 \text{ kN}$  is given by

$$U_0 = \frac{P_0^2}{2k} = \frac{(50 \times 10^3)^2}{2 \times [(210 \times 10^9 \times 20 \times 10 \times 10^{-6})/1]} = 29.76 \text{ J} \quad (16)$$

Now, suppose that the bar has been segmented into two bars with cross-sections  $10 \text{ mm} \times 10 \text{ mm}$  and length  $1 \text{ m}$ , and the force  $P_0$  has been segmented into two *unequal* forces with magnitudes  $40$  and  $10 \text{ kN}$ , applied to the individual bars. In this case, the accumulated elastic strain energy in the bars is

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{(40 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6}/1]} + \frac{(10 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6}/1]} = 38.1 + 2.38 = 40.48 \text{ J} \quad (17)$$

This is about 36% larger than the elastic strain energy of  $29.76 \text{ J}$  accumulated in the single bar.

Increasing the capability of accumulating elastic strain energy is important not only to prevent failure during dynamic loading but also in cases where more elastic strain energy needs to be stored.

If the bar had been segmented into two bars with cross sections  $8 \text{ mm} \times 10 \text{ mm}$  and  $12 \text{ mm} \times 10 \text{ mm}$  and the force had been segmented into two forces with magnitudes  $20$  and  $30 \text{ kN}$ , the accumulated elastic strain energy characterising the segmented bar and loading would be equal to the elastic more elastic strain energy characterising the original bar and loading:

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{(20 \times 10^3)^2}{2 \times [210 \times 10^9 \times 8 \times 10 \times 10^{-6}/1]} + \frac{(30 \times 10^3)^2}{2 \times [210 \times 10^9 \times 12 \times 10 \times 10^{-6}/1]} = 29.76 \text{ J}$$

This is because the displacements of the segmented bars are equal (the ratios  $P_1/k_1 = P_2/k_2 = \delta$ , are the same) and, according to the basic properties of inequality (15), in this case, equality is attained.

If the bar had been segmented into two bars with cross sections  $10 \text{ mm} \times 10 \text{ mm}$  and the force had been segmented into two equal forces with magnitude  $25 \text{ kN}$ , again, the accumulated strain energy in the bar would be the same:

$$U_1 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = 2 \times \frac{(25 \times 10^3)^2}{2 \times [210 \times 10^9 \times 10 \times 10 \times 10^{-6}/1]} = 29.76 \text{ J}$$

because the displacements of the bars  $P_1/k_1 = P_2/k_2 = \delta$  are the same.

In order to obtain any advantage from inequality (15), *asymmetry must be present in the system so that the displacements of the bars are not equal*:  $\delta_1 = P_1/k_1 \neq \delta_2 = P_2/k_2$ . Existence of asymmetry is absolutely essential for increasing system's performance through segmentation based on inequality (15). Segmented bars experiencing the same displacement do not yield an increase of the amount of stored elastic strain energy.

The asymmetry requirement to achieve increased energy storage is rather counterintuitive and makes this result difficult to obtain by alternative means bypassing inequality (11).

Consider the sequence  $\{P_1^2, P_2^2, \dots, P_n^2\}$  and the sequence  $\{1/(2k_1), 1/(2k_2), \dots, 1/(2k_n)\}$ . The dot product of these sequences is the right-hand side of inequality (15). According to the rearrangement inequality, the dot product of two sequences is maximum if they are similarly ordered, for example if  $P_1^2 \leq P_2^2 \leq \dots \leq P_n^2$  and  $1/(2k_1) \leq 1/(2k_2) \leq \dots \leq 1/(2k_n)$ . For the first sequence, it can be shown that if  $P_1^2 \leq P_2^2 \leq \dots \leq P_n^2$  then  $P_1 \leq P_2 \leq \dots \leq P_n$ . Indeed, from the basic properties of inequalities, for positive  $P_i$  and  $P_j$  from  $P_i^2 \leq P_j^2$ , it follows that  $P_i \leq P_j$ .

For the second sequence, it is easy to see that  $1/k_1 \leq 1/k_2 \leq \dots \leq 1/k_n$  only if  $k_1 \geq k_2 \geq \dots \geq k_n$ . For any other permutation of the sequences, for example  $\{P_{k_1}, P_{k_2}, \dots, P_{k_n}\}$  and  $\{k_{s_1}, k_{s_2}, \dots, k_{s_n}\}$ , the next inequality is fulfilled:

$$\frac{P_1^2}{2k_1} + \dots + \frac{P_n^2}{2k_n} \geq \frac{P_{k_1}^2}{2k_{s_1}} + \dots + \frac{P_{k_n}^2}{2k_{s_n}}$$

As a result, the right-hand side of inequality (15) is maximised if the ordered in ascending order segmented loads  $P_1 \leq P_2 \leq \dots \leq P_n$  are paired with the ordered in descending order stiffness values:  $k_1 \geq k_2 \geq \dots \geq k_n$ .

**Increasing the accumulated strain energy for components loaded in bending**

An alternative interpretation of inequality (11) can be created if variable  $a$ , for example, stands for the additive quantity ‘force’ and variable  $b$  stands for the additive quantity ‘flexural stiffness’.

It is a well-known result from mechanics of materials that the accumulated elastic strain energy  $U$  in a cantilever beam (Figure 3(a)) with rectangular cross section  $b \times h$  and length  $L$  is given by the equation:

$$U = \frac{1}{2} P \times f \tag{18}$$

where  $P$  [N] denotes the loading force and  $f$  [m] is the elastic deflection at the point of application of the concentrated force  $P$ .

From mechanics of materials,<sup>31,39</sup> for a cantilever beam with rectangular cross-section, the deflection  $f$  can be determined from

$$f = \frac{PL^3}{3EI} \tag{19}$$

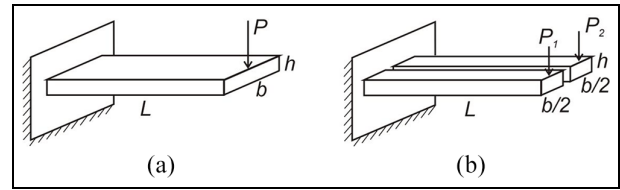
where  $E$  is the Young modulus of the material and  $I = bh^3/12$  is the second moment of area of the beam. Substituting  $I$  in (19) gives

$$f = \frac{4PL^3}{Ebh^3} \tag{20}$$

According to equation (18), the accumulated elastic strain energy  $U$  in the cantilever beam is then given by

$$U = \frac{2P^2L^3}{Ebh^3} \tag{21}$$

By introducing the value  $k = \frac{Ebh^3}{4L^3}$ , standing for the flexural stiffness of the cantilever beam, equation (18) for the accumulated elastic strain energy can be written as



**Figure 3.** (a) Cantilever beam loaded with a single concentrated force  $P$  and (b) segmented cantilever beam loaded with two forces  $P_1$  and  $P_2$  whose sum is equal to the original force  $P$  (the segmented beams are loaded independently and do not necessarily have the same displacements and stresses).

$$U = \frac{P^2}{2k} \tag{22}$$

Suppose that the load  $P$  and the cantilever beam have been segmented into  $n$  loads  $P_1, P_2, \dots, P_n$  ( $\sum_{i=1}^n P_i = P$ ) and  $n$  beams with the same thickness  $h$  and smaller widths  $b_1, b_2, \dots, b_n$  ( $\sum_{i=1}^n b_i = b$ ). The flexural stiffness  $k$  of the original beam is then equal to the sum of the flexural stiffness values characterising the smaller cantilever beams:  $k = k_1 + k_2 + \dots + k_n$ , where  $k_i = \frac{Eb_i h^3}{4L^3}$ .

According to inequality (11),

$$\frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} + \dots + \frac{P_n^2}{2k_n} \geq \frac{(P_1 + P_2 + \dots + P_n)^2}{2(k_1 + k_2 + \dots + k_n)} \tag{23}$$

The expression  $\frac{(P_1 + P_2 + \dots + P_n)^2}{2(k_1 + k_2 + \dots + k_n)}$  in the right-hand side of (23) can be interpreted as the accumulated elastic strain energy due to a load  $P$  acting on a cantilever beam with flexural stiffness  $k = k_1 + k_2 + \dots + k_n$  (the design option in Figure 3a). The left-hand side of (23) can be interpreted as the accumulated elastic strain energy from loads  $P_i$  ( $\sum_i P_i = P$ ), resulting from segmenting the original load  $P$  into smaller loads and applying the smaller loads to the smaller cantilever beams with flexural stiffness values  $k_i$  (the design option in Figure 3b).

Inequality (23) then predicts that *the accumulated elastic strain energy due to a load applied on a cantilever beam, is smaller than the accumulated elastic strain energy from loads resulting from segmenting the original load into smaller loads, and applying the smaller loads on the cantilever beam segments.*

Similar to the previous examples related to bars loaded in tension, not every segmentation of the beam and the load leads to an increase of the accumulated elastic strain energy. If the smaller cantilever beams have the same displacement under their loads, no increase in the accumulated strain energy will be present. Indeed, in inequality (23), the value  $f_i = P_i/k_i$  is the displacement of the  $i$ th cantilever beam under load  $P_i$ . If all beams displacements are equal

( $f_1 = f_2 = \dots = f_n = f$ ), equality is attained in (23). Indeed, the left-hand side of (23) gives  $\frac{P_1^2}{2k_1} + \dots + \frac{P_n^2}{2k_n} = (f/2)(P_1 + P_2 + \dots + P_n) = (f/2) \times P$ . Since,  $P/(k_1 + \dots + k_n) = f$ , the right-hand side of (23) gives  $P^2/[2(k_1 + \dots + k_n)] = (f/2) \times P$  hence, equality is indeed attained in equation (23).

Again, the requirement for asymmetry to achieve the effect of increased elastic strain energy is rather counterintuitive and makes this result difficult to obtain by alternative means, bypassing the use of inequality (11).

The meaningful interpretation of the abstract inequality (11) helped to find overlooked properties in the mature field of mechanical engineering. No such results have been reported in modern comprehensive textbooks in mechanical engineering and stress analysis<sup>23–30,39</sup>

By interpreting inequalities, new results can always be obtained, without the need of any forward analysis. The generated new knowledge can then be used for optimising various systems and processes in any area of science and technology.

### Constructing a system with superior reliability

Often, additional manipulation must be applied in order to make the algebraic inequality interpretable. Consider the non-trivial algebraic inequality

$$(1 - a^2b)(1 - b^2c)(1 - c^2a) \geq (1 - a^3)(1 - b^3)(1 - c^3) \quad (24)$$

where  $a, b, c$  are positive numbers smaller than 1. This inequality can be proved by combining the methods of symmetry and the AM-GM inequality.

Using the symmetry of the inequality, without loss of generality,  $a \geq b \geq c \geq 0$  can be assumed.

Inequality (24) can be proved if the equivalent inequality

$$\sqrt[3]{\frac{1 - a^3}{1 - a^2b} \times \frac{1 - b^3}{1 - b^2c} \times \frac{1 - c^3}{1 - c^2a}} \leq 1 \quad (25)$$

can be proved.

From the AM-GM inequality, it follows that

$$\begin{aligned} & \sqrt[3]{\frac{1 - a^3}{1 - a^2b} \times \frac{1 - b^3}{1 - b^2c} \times \frac{1 - c^3}{1 - c^2a}} \\ & \leq \frac{1}{3} \left( \frac{1 - a^3}{1 - a^2b} + \frac{1 - b^3}{1 - b^2c} + \frac{1 - c^3}{1 - c^2a} \right) \end{aligned}$$

Consequently, if inequality (26)

$$\frac{1}{3} \left( \frac{1 - a^3}{1 - a^2b} + \frac{1 - b^3}{1 - b^2c} + \frac{1 - c^3}{1 - c^2a} \right) \leq 1 \quad (26)$$

can be proved, inequalities (25) and (24) will follow.

Proving inequality (26) is reduced to proving the equivalent inequality

$$\frac{1 - a^3}{1 - a^2b} + \frac{1 - b^3}{1 - b^2c} + \frac{1 - c^3}{1 - c^2a} \leq 1 + 1 + 1$$

which is equivalent to the inequality

$$\frac{a^2(a - b)}{1 - a^2b} + \frac{b^2(b - c)}{1 - b^2c} + \frac{c^2(c - a)}{1 - c^2a} \geq 0 \quad (27)$$

To prove inequality (27), the techniques 'introducing mutually exclusive cases' and 'proving an intermediate inequality' will be applied.

Consider two mutually exclusive cases, related to the variables  $a, b$  and  $c$ : (i)  $b^2 \leq ac$  and (ii)  $b^2 > ac$ .

Let  $b^2 \leq ac$  hold (case i). In this case,  $b^2c \leq ac^2$  also holds and the left side of (27) will be decreased if the denominator  $1 - a^2b$  in the first term of (27) is replaced by the larger value  $1 - b^2c$ , the denominator  $1 - c^2a$  of the third term is replaced by the larger value  $1 - b^2c$  and  $b^2$  in the numerator of the second term is replaced by the smaller factor  $c^2$  ( $c < b$ ). As a result,

$$\begin{aligned} & \frac{a^2(a - b)}{1 - a^2b} + \frac{b^2(b - c)}{1 - b^2c} + \frac{c^2(c - a)}{1 - c^2a} \geq \frac{a^2(a - b)}{1 - b^2c} \\ & + \frac{c^2(b - c)}{1 - b^2c} + \frac{c^2(c - a)}{1 - b^2c} \end{aligned}$$

holds. Factoring  $c^2$  from the second and third term of the right hand side of the last inequality results in:

$$\begin{aligned} & \frac{a^2(a - b)}{1 - b^2c} + \frac{c^2(b - c)}{1 - b^2c} + \frac{c^2(c - a)}{1 - b^2c} = \frac{a^2(a - b)}{1 - b^2c} \\ & + \frac{c^2(b - a)}{1 - b^2c} = (a - b) \frac{a^2 - c^2}{1 - b^2c} \end{aligned} \quad (28)$$

for the right-hand side.

Since  $a^2 - c^2 \geq 0$ ,  $a - b \geq 0$  and  $1 - b^2c \geq 0$ , the right-hand side of (28) is greater than zero or equal to zero. The intermediate inequality  $(a - b) \frac{a^2 - c^2}{1 - b^2c} \geq 0$  is true, which shows that if case (i) holds, inequality (27) and the original inequality (24) also hold.

Now let  $b^2 > ac$  hold (case ii). In this case,  $b^2c > ac^2$  also holds and the left side of (27) will be decreased if the denominator  $1 - a^2b$  in the first term is replaced by the larger value  $1 - c^2a$ , the denominator  $1 - b^2c$  of the second term is replaced by the larger value  $1 - c^2a$  and  $b^2$  in the numerator of the second term is replaced by the smaller factor  $c^2$ . As a result,

$$\begin{aligned} & \frac{a^2(a - b)}{1 - a^2b} + \frac{b^2(b - c)}{1 - b^2c} + \frac{c^2(c - a)}{1 - c^2a} \\ & > \frac{a^2(a - b)}{1 - c^2a} + \frac{c^2(b - c)}{1 - c^2a} + \frac{c^2(c - a)}{1 - c^2a} \end{aligned} \quad (29)$$

holds. Factoring  $c^2$  from the second and third term of the right-hand side of inequality (29) results in:

$$\frac{a^2(a-b)}{1-c^2a} + \frac{c^2(b-c)}{1-c^2a} + \frac{c^2(c-a)}{1-c^2a} = \frac{a^2(a-b)}{1-c^2a} + \frac{c^2(b-a)}{1-c^2a} = (a-b)\frac{a^2-c^2}{1-c^2a}$$

Since  $a^2 - c^2 \geq 0$ ,  $a - b \geq 0$  and  $1 - c^2a \geq 0$ , it follows that  $(a - b)\frac{a^2 - c^2}{1 - c^2a} \geq 0$ . This shows that the left-hand side of inequality (29) is non-negative which proves inequality (27) and the original inequality (24). The proofs of the two mutually exclusive and exhaustive cases (i) and (ii) prove inequality (24).

Multiplying the left and right part of (24) by ‘-1’ reverses the direction of the inequality and adding ‘1’ to the left and right side of inequality (24) yields the inequality

$$1 - (1 - a^2b)(1 - b^2c)(1 - c^2a) \leq 1 - (1 - a^3)(1 - b^3)(1 - c^3) \tag{30}$$

An interpretation can now be given to the resultant inequality (30) which is equivalent to inequality (24).

Consider the system configurations in Figure 4 built on three different types components *A*, *B* and *C*. A system from Figure 4 is in working state if there is path through working components from any of the source nodes  $s_1, s_2, s_3$  to the end node CR. In other words, all three components from at least one parallel line must be in working state for the system to be in working state.

Systems with parallel-series arrangements of the type in Figure 4 are very common. Lines cooling a reactor by working in parallel are examples of such systems. In a cooling line, the building sections are logically arranged in series (which means that a cooling line is working if all sections of the cooling line are in working state). The chemical reactor is adequately cooled if at least a single cooling line is in working state and delivers cooling fluid from any of the sources  $s_1, s_2$  and  $s_3$  to the chemical reactor CR. As a result, with respect to cooling the reactor CR, the cooling lines are arranged in parallel.

Now suppose that each cooling line contains three sections, each of which can be of different type: type A, type B or type C. Let  $a, b$  and  $c$  in (30) stand for the reliabilities of components of types *A*, *B* and *C* in the system configurations from Figure 4. The left-hand side of inequality (30) now represents the reliability of the system configuration in Figure 4(a) while the right-hand side of inequality (30) represents the reliability of the system configuration in Figure 4(b).

Inequality (30) predicts that the reliability of the system configuration in Figure 4(b) is greater than the reliability of the system configuration in Figure 4(a).

The difference in the reliabilities of the two competing system configurations is significant and this can be seen after substituting specific values, for example,  $a = 0.8, b = 0.6$  and  $c = 0.2$ , for the reliabilities of components. From the right-hand side of inequality (30), the value

$$R_b = 1 - (1 - a^3)(1 - b^3)(1 - c^3) = 0.62$$

is obtained for the reliability of the system configuration in Figure 4(b) while from the left-hand side of inequality (30), the value

$$R_a = 1 - (1 - a^2b)(1 - b^2c)(1 - c^2a) = 0.45$$

is obtained for the reliability of the system configuration in Figure 4(a).

In a similar fashion, it can be shown that the inequalities

$$1 - (1 - a^2c)(1 - b^2a)(1 - c^2b) \leq 1 - (1 - a^3)(1 - b^3)(1 - c^3) \tag{31}$$

$$1 - (1 - abc)(1 - c^2b)(1 - a^2b) \leq 1 - (1 - a^3)(1 - b^3)(1 - c^3) \tag{32}$$

$$1 - (1 - abc)(1 - a^2c)(1 - b^2c) \leq 1 - (1 - a^3)(1 - b^3)(1 - c^3) \tag{33}$$

$$1 - (1 - abc)(1 - abc)(1 - abc) \leq 1 - (1 - a^3)(1 - b^3)(1 - c^3) \tag{34}$$

are also true.

The left-hand side of inequalities (31–34) is the reliability of the system configurations in Figure 4(c) to (f), correspondingly, while the right-hand side of inequalities (31–34) is the reliability of the system in Figure 4(b).

As a result, a general conclusion can be made that the parallel-series system configuration in Figure 4(b) is intrinsically more reliable than the system configurations in Figures 4(a), (c) to (f), composed of the same components.

For example, for a cooling system composed of three new sections *A*, three medium-age sections *B* and three old sections *C*, arranging all new sections *A* in the same cooling branch, all medium sections *B* in another cooling branch and all old sections *C* in a separate cooling branch (Figure 4(b)) maximises the reliability of the system.

Inequality (30) can be generalised easily for more than three types of components in a branch.

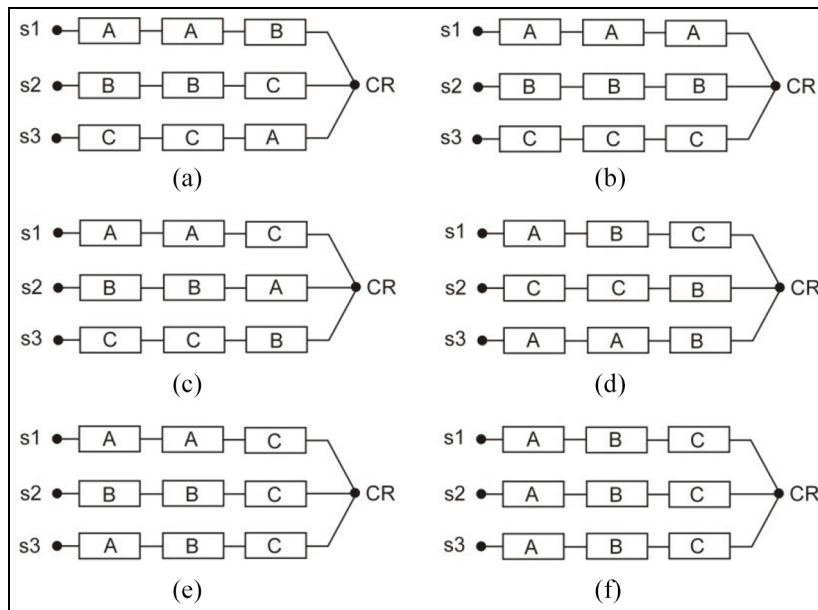


Figure 4. A parallelseries cooling system with different system configurations: (a), (b), (c), (d), (e) and (f).

**Increasing the probability of successful accomplishment of tasks by devices with unknown reliability**

The approach based on meaningful interpretation of algebraic inequalities can also be applied to ranking alternative decisions. Consider the non-trivial abstract inequalities:

$$x^3 + y^3 + z^3 \geq 3xyz \tag{35}$$

$$2(x^3 + y^3 + z^3) \geq x^2y + x^2z + y^2x + y^2z + z^2x + z^2y \tag{36}$$

Both inequalities can be proved by invoking the Muirhead's inequality.<sup>4</sup>

The Muirhead's inequality states that if the sequence {a} is majorising the sequence {b} and  $x_1, x_2, \dots, x_n$  are non-negative, the next inequality holds:

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \tag{37}$$

Consider two non-increasing sequences  $a_1 \geq a_2 \geq \dots \geq a_n$  and  $b_1 \geq b_2 \geq \dots \geq b_n$  of non-negative real numbers. The sequence {a} is said to majorise the sequence {b} if the following conditions are fulfilled:

$$\begin{aligned} a_1 &\geq b_1; a_1 + a_2 \geq b_1 + b_2; \dots; a_1 + a_2 + \dots + a_{n-1} \\ &\geq b_1 + b_2 + \dots + b_{n-1}; a_1 + a_2 + \dots + a_{n-1} \\ &+ a_n = b_1 + b_2 + \dots + b_{n-1} + b_n \end{aligned} \tag{38}$$

For any set of non-negative numbers  $x_1, x_2, \dots, x_n$ , a symmetric sum is defined as  $\sum_{sym} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$  which, when expanded, includes  $n!$  terms. Each term is formed by a distinct permutation of the elements of the sequence  $a_1, a_2, \dots, a_n$ . Thus, if  $\{a\} = [2, 1, 0]$  then

$$\sum_{sym} x_1^2 x_2^1 x_3^0 = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$$

If  $\{a\} = [2, 0, 0]$ , then

$$\sum_{sym} x_1^2 x_2^0 x_3^0 = 2x_1^2 + 2x_2^2 + 2x_3^2$$

Consider now the sequences  $\{a\} = [3, 0, 0]$  and  $\{b\} = [1, 1, 1]$ . Because the sequence  $\{a\} = [3, 0, 0]$  majorises the sequence  $\{b\} = [1, 1, 1]$ , inequality (39) is obtained.

$$(n - 1)! \times (x^3 + y^3 + z^3) \geq n! \times xyz \tag{39}$$

where  $n = 3$ . By dividing both sides of (39) to  $(n - 1)!$ , inequality (39) transforms into inequality (35).

Since the sequence  $\{a\} = [3, 0, 0]$  also majorises the sequence  $\{c\} = [2, 1, 0]$ , the following inequality also follows immediately from the Muirhead's inequality (37):

$$\begin{aligned} (n - 1)! \times (x^3 + y^3 + z^3) &\geq x^2y + x^2z + y^2x + y^2z \\ &+ z^2x + z^2y \end{aligned} \tag{40}$$

which, for  $n = 3$  gives inequality (36).

If the left and the right hand side of inequality (35) are multiplied by  $1/3$ , the inequality

$$\frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 \geq xyz \tag{41}$$

is obtained, which can be interpreted by using the total probability theorem for mutually exclusive events. The factors 1/3 on the left hand side of (41) can be interpreted as probabilities of selecting any of the three arrangements (Figure 5(a), arrangements 1, 2 and 3). Each arrangement includes three devices of types X, Y and Z assigned to a mission including three identical tasks. The probabilities x, y and z of successful accomplishment of a task by a device of type X, Y and Z, are unknown.

A mission is considered to have been accomplished successfully, if all three tasks assigned to the devices have been accomplished successfully.

According to the total probability theorem, the left-hand side of inequality (41) represents the total probability that a randomly selected arrangement consisting of three devices of the same type, allocated to three identical tasks (Figure 5(a)), will successfully accomplish a mission.

Indeed, a successful accomplishment of a mission by devices of the same type can occur in three different, mutually exclusive ways: (i) arrangement 1, including three devices of type X is randomly selected and all devices successfully accomplish their tasks, the probability of which is (1/3)x<sup>3</sup>; (ii) arrangement 2 including devices of type Y is randomly selected and all devices successfully accomplish their tasks, the probability of which is (1/3)y<sup>3</sup>; and finally, (iii) arrangement 3 including devices of type Z is randomly selected and all devices successfully accomplish their tasks, the probability of which is (1/3)z<sup>3</sup>.

The right-hand side of inequality (41) is the probability xyz that the three devices of different types (X, Y and Z) successfully accomplish their tasks.

If the left- and the right-hand side of inequality (36) are multiplied by 1/6, the inequality

$$\begin{aligned} & \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 \geq \\ & \frac{1}{6}x^2y + \frac{1}{6}x^2z + \frac{1}{6}y^2x + \frac{1}{6}y^2z \\ & + \frac{1}{6}z^2x + \frac{1}{6}z^2y \end{aligned} \tag{42}$$

is obtained, which can be meaningfully interpreted again by using the total probability theorem for mutually exclusive events. According to the total probability theorem, the left-hand side of inequality (42) represents the total probability that a randomly selected arrangement consisting of three identical-type devices will successfully accomplish the mission (Figure 6(a)).

The right-hand side of inequality (42) represents the total probability that a randomly selected arrangement composed of three devices, two of which are of the same type, will successfully accomplish the mission. A successful accomplishment of a mission including three

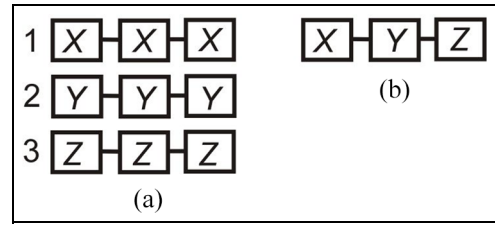


Figure 5. (a) Selecting an arrangement of three devices allocated to three identical tasks and (b) an arrangement that includes three devices of different type assigned to three identical tasks.

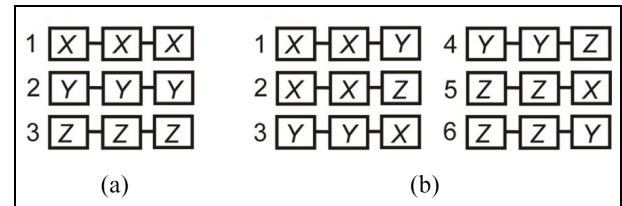


Figure 6. (a) Selecting an arrangement that includes three devices of the same type and (b) selecting an arrangement that includes three devices two of which are of the same type.

devices, two of which are of the same type, can occur in six distinct ways (Figure 6(b)): (i) the device arrangement (X, X and Y) is randomly selected, and all three tasks are successfully accomplished by the devices, the probability of which is (1/6)x<sup>2</sup>y; (ii) the device arrangement (X, X and Z) is randomly selected, and all three tasks are successfully accomplished by the devices, the probability of which is (1/6)x<sup>2</sup>z and so on.

Despite that the probabilities x, y and z of successful accomplishment of a task by the devices of different types are unknown, according to the predictions of inequalities (41) and (42), the strategy of randomly selecting an arrangement composed of three devices of the same type is characterised by a higher chance of successful accomplishment of the mission compared to selecting an arrangement composed of three different types of devices or an arrangement composed of three devices two of which are of the same type.

This is a counter-intuitive result considering that the probabilities x, y and z of successful accomplishment of a task by the different types of devices are unknown. Why should a particular strategy provide an advantage, if the probabilities x, y and z are unknown? It seems that in such a situation of deep uncertainty, no particular strategy matters, yet this impression is wrong. Despite the existing deep uncertainty related to the probabilities of successful accomplishment of the tasks by the devices, selecting randomly an arrangement composed of the same type devices is always the best strategy. This conclusion holds *irrespective of existing unknown interdependencies among the probabilities of successful accomplishment of the tasks from the devices.*

The significant advantage provided by the superior strategy can be illustrated by a numerical example.

Suppose that the probabilities of accomplishing the tasks characterising the different device types  $X$ ,  $Y$  and  $Z$  are  $x = 0.88$ ,  $y = 0.64$  and  $z = 0.38$ . According to inequality (41), the probability that a randomly selected arrangement including three devices of the same type will successfully accomplish the mission is:

$$p_1 = (1/3) \times 0.88^3 + (1/3) \times 0.64^3 + (1/3) \times 0.38^3 = 0.33$$

while the probability of accomplishing successfully the mission by three devices from different types is  $p_2 = 0.88 \times 0.64 \times 0.38 = 0.21$ .

According to inequality (42), the probability of accomplishing the mission by a randomly selected arrangement consisting of three devices two of which are of the same type is:

$$p_3 = (1/6) \times [0.88^2 \times 0.64 + 0.88^2 \times 0.38 + 0.64^2 \times 0.88 + 0.64^2 \times 0.38 + 0.38^2 \times 0.88 + 0.38^2 \times 0.64] = 0.25$$

By using the Muirhead's inequality for  $n > 3$ , these results are naturally generalised for  $n > 3$  tasks composing the mission (the generalisation is straightforward and to conserve space, details have been omitted). The strategy of selecting randomly an arrangement composed of  $n$  devices of the same type is characterised by the highest chance of successful accomplishment of the mission.

The predictions from the inequalities of this paper have been confirmed by Monte Carlo simulations, each of which involved 10 million trials. The method based on meaningful interpretation of algebraic inequalities is truly domain-independent; it works in any domain of human activity. The presented results have never been suggested in the literature which indicates that the lack of knowledge of the method of algebraic inequalities made these results invisible to domain experts.

### Avoiding overestimation of the expected return from investments by a meaningful interpretation of the Chebyshev's sum inequality

Algebraic inequalities can be interpreted to avoid an overestimation or underestimation of a particular factor. This application will be illustrated with the Chebyshev's sum inequality.

#### Chebyshev's sum inequality

Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be two sets of similarly ordered positive numbers ( $a_1 \leq \dots \leq a_n$  and  $b_1 \leq \dots \leq b_n$ ). Then the following classical inequality holds:<sup>35</sup>

$$\frac{a_1 b_1 + \dots + a_n b_n}{n} \geq \frac{a_1 + \dots + a_n}{n} \cdot \frac{b_1 + \dots + b_n}{n} \quad (43)$$

If the number sequences are oppositely ordered, for example, if  $a_1 \geq \dots \geq a_n$  and  $b_1 \leq \dots \leq b_n$ , the inequality is reversed.

$$\frac{a_1 b_1 + \dots + a_n b_n}{n} \leq \frac{a_1 + \dots + a_n}{n} \cdot \frac{b_1 + \dots + b_n}{n} \quad (44)$$

Chebyshev's sum inequality possesses a great advantage. It provides the opportunity to segment an initial complex expression into simpler expression. Thus, the  $a_i b_i$  terms in (43) and (44) are segmented into simpler terms involving  $a_i$  and  $b_i$ , only.

The segmentation capability provided by the Chebyshev's sum inequality will be illustrated with an example related to the risk of overestimating the expected potential return from risky investments by using average values.

Suppose that in inequality (44), the variables the variables  $a_i$  ( $i = 1, \dots, n$ ), ranked in descending order ( $a_1 \geq \dots \geq a_n$ ) stand for probabilities of a return from risky investments. The variables  $b_i$  ( $i = 1, \dots, n$ ), ranked in ascending order ( $b_1 \leq \dots \leq b_n$ ), stand for the magnitudes of the returns which correspond to the probabilities  $a_i$ . This negative correlation between the probabilities and returns from risky investments is *very common*: the smaller the likelihood of success characterising a risky investment, the larger is the return from the investment.

The right-hand side of inequality (44) can now be interpreted as an estimate of the expected profit per investment by using the average return from an investment  $\bar{b} = (b_1 + b_2 + \dots + b_n)/n$  and the average probability of return from an investment  $\bar{a} = (1/n)(a_1 + \dots + a_n)$ . The left-hand side of inequality (44) can be interpreted as the expected profit per investment, assessed by taking the expected profit from each individual investment.

Inequality (44) predicts that estimating the expected return per investment, by using the average return and the average probability of return from an investment, leads to an overestimation of the expected return per investment.

The overestimation of the expected potential return from an investment can be significant and this will be illustrated by a very simple numerical example involving only two investments: an investment with potential return of  $c_1 = \$800$  and an investment with potential return of  $c_2 = \$15000$ . The probability of return from the investments are  $p_1 = 0.77$  and  $p_2 = 0.60$ , correspondingly. If the average probability of return from an investment and the average magnitude of the return are used for calculating the expected potential return, the value  $\bar{p} \times \bar{c} = 0.5(0.77 + 0.60) \times 0.5(800 + 15000)$

= 5411 for the expected potential return per investment will be predicted. The actual expected potential return per investment is

$$\frac{p_1c_1 + p_2c_2}{2} = \frac{0.77 \times 800 + 0.60 \times 15000}{2} = 4808$$

which is significantly smaller than the prediction of 5411, based on the average values of the probabilities and the returns.

## Conclusions

1. Inequalities based on sub-additive multivariable functions can be interpreted meaningfully if their variables and separate terms represent additive quantities. The generated new knowledge can be used for optimising systems and processes in diverse areas of science and technology.
2. Meaningful interpretation of the Bergström inequality, which is based on a multivariate sub-additive function, can be used to increase the accumulated strain energy in components loaded in tension and bending. For components loaded with a single external force, the amount of stored elastic strain energy can be increased by segmenting the component and the external force and applying the smaller forces to the segmented components.
3. Existence of asymmetry is essential for increasing the system's performance by using the Bergström inequality. Segmented components which experience the same displacement do not yield an increase of the amount of stored elastic strain energy. The requirement for asymmetry to achieve beneficial effects is counterintuitive and makes these results difficult to obtain by alternative methods bypassing the use of inequalities.
4. The meaningful interpretation of non-trivial algebraic inequalities yielded a highly counter-intuitive result related to assigning devices of different types to missions involving identical tasks. In the case where the probabilities of successful accomplishment of a task characterising the devices are unknown, the best strategy for successful accomplishment of a mission including several identical tasks consists of selecting randomly an arrangement including devices of the same type. This is always the best strategy irrespective of existing unknown interdependencies among the probabilities of successful accomplishment of the tasks, characterising the devices.
5. An interpretation of a non-trivial algebraic inequality has been used to construct the most reliable parallel-series system and reduce the risk of system failure.

6. An interpretation of the Chebyshev's sum inequality can be used to avoid an overestimation of average potential returns from investments.


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