A GENERAL CLASS OF ALGEBRAIC INEQUALITIES FOR GENERATING NEW KNOWLEDGE AND OPTIMISING THE DESIGN OF SYSTEMS AND PROCESSES

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Abstract

A special class of general inequalities has been identified that provides the opportunity for generating new knowledge that can be used for optimising systems and processes in diverse areas of science and technology. It is demonstrated that inequalities belonging to this class can always be interpreted meaningfully if the variables and separate terms of the inequalities represent additive quantities.

The meaningful interpretation of a new algebraic inequality based on the proposed general class of inequalities led to developing a light-weight design for a supporting structure based on cantilever beams, reducing the maximum force upon impact, generating new knowledge about the deflection of elastic elements connected in parallel and series and optimising the allocation of resources to maximise the expected benefit. The interpretation of the new inequality yielded the result stating that the deflection of elastic elements connected in parallel is at least n^2 times smaller than the deflection of the same elastic elements connected in series irrespective of the individual stiffness values of the elastic elements.

The interpretation of another algebraic inequality from the proposed general class led to a method for decreasing the stiffness of a mechanical assembly by cyclic permutation of the elastic elements building the assembly. The analysis showed that a decrease of stiffness exists only if asymmetry of the stiffness values in the connected elements is present.

Keywords: *algebraic inequalities; interpretation of algebraic inequalities, light-weight design; maximum force during impact; equivalent stiffness*

1. Introduction

Algebraic inequalities have been used extensively in mathematics and a number of useful non-trivial algebraic inequalities and their properties have been well documented (Fink 2000; Bechenbach and Bellman 1961; Engel 1998; Hardy et al. 1999; Pashpatte 2005; Cvetkovski 2012; Marshall et al. 2010; Steele 2004; Kazarinoff 1961; Sedrakyan and Sedrakyan 2010).

In mathematics, for a long time, simple inequalities are being used to express error bounds in approximations and constraints in linear programming models. In physics and engineering, applications of inequalities have also been considered (Cloud et al. 2014; Samuel and Weir 1999; Rastegin 2012). In engineering design, design inequalities have been widely used to express design constraints guaranteeing that the design will perform its required function. More recently, in non-linear programming problems, systems of inequalities have been used to present, non-linear goal and non-linear resource constraints (Corley and Dwobeng 2020). Inequalities have also been used in reliability and risk research to characterise reliability functions (Ebeling 1997; Xie and Lai 1998; Makri and Psilakis 1996; Hill et al. 2013; Berg and Kesten 1985; Kundu and Ghosh 2017; Dohmen 2006).

Algebraic inequalities are particularly suitable for handling unstructured uncertainty. Most of the conventional approaches deal with structured uncertainty. Handling uncertainty is usually done through probabilities assessed by using a data-driven approach or by using the Bayesian, subjective probability approach. A major deficiency of the data-driven approach is that probabilities cannot always be meaningfully defined. These deficiencies cannot be rectified by using the Bayesian approach which is not so critically dependent on the availability of past failure data because it uses assigned subjective probabilities. The Bayesian approach however, depends on a selected probability model that may not be relevant to the modelled phenomenon/process (Aven 2017). In addition, the assigned subjective probabilities depend on the available knowledge and vary significantly among the assessors. Furthermore, weak background knowledge at the basis of the assigned subjective probabilities often results in poor predictions.

There are approaches in decision-making which deal with unstructured uncertainty (Ben-Haim 2005; Schor and Reich 2003). These approaches are based on the info-gap model of uncertainty and do not require probability distributions. Although the info-gap theory (Ben-Haim 2005) deals with unstructured uncertainty by not making probability distribution assumptions, it still requires assumptions to be made about designer's best estimate.

Single-objective optimisation strategies in engineering revolve around determining the optimum of an objective function, commonly by using an exact analytical method or an heuristic algorithm such as a genetic algorithm (Wei et al. 2021). In multi-objective optimization strategies, the best compromise among possible designs is found. When the objectives are functions of a single design variable, Pareto-optimal solutions are sought (Serhat and Basdogan 2019). In both cases however, in order to formulate the objective function, the values of the controlling parameter entering the objective function must be known.

In contrast, algebraic inequalities do not require the distributions or the values of the variables entering the inequalities or any other assumptions. This advantage permits the use of algebraic inequalities for handling deep unstructured uncertainty (Todinov 2020a, 2020b). In this respect, algebraic inequalities avoid a major difficulty in most of the conventional models for handling uncertainty – lack of meaningful specification of frequentist probabilities or lack of justification behind the assigned subjective probabilities and probabilistic models.

Thus, by using algebraic inequalities, two systems can be compared without any knowledge of the reliabilities of their components (Todinov 2020a). Algebraic inequalities also provide a strong support for risk-critical decisions under deep uncertainty. Although the distributions of the risk-critical parameters remain unknown, the method of algebraic inequalities can still establish the intrinsic superiority of one of the competing options.

Suppose that a particular system/process, performing a certain function can be developed in two different configurations and that the two sides of an algebraic inequality represent the outputs related to these two configurations. Unlike algebraic equalities, which establish that two different configurations (states) of a system or a process are equivalent, the algebraic inequality establishes that one of the compared configurations (states) is superior. For this reason, the algebraic inequalities open opportunities for enhancing the performance of systems and processes.

The forward approach to using algebraic inequalities (Todinov 2020a) includes the following steps: (i) detailed analysis of the system or process, (ii) conjecturing algebraic inequalities ranking the competing alternatives, (iii) testing the conjectured inequality by using Monte Carlo simulation and (iv) proving the conjectured inequality rigorously. This way of exploiting algebraic inequalities has already been demonstrated (Todinov 2020a, 2020c) with comparing systems with unknown reliabilities of their components.

This paper demonstrates that the use of algebraic inequalities in engineering is far reaching and is certainly not restricted to specifying design constraints. Algebraic inequalities also admit meaningful interpretation that can be used to generate new knowledge related to a real system or process. This is a new dimension in the use of algebraic inequalities and another formidable advantage. Depending on the specific interpretation, knowledge applicable to different systems from different domains can be released from the same inequality. This is the essence of inverse approach to using algebraic inequalities (Figure 1b). In contrast to the forward approach, the inverse approach moves in the opposite direction. It starts with a correct algebraic inequality, continues with creating relevant meaning for the variables entering the inequality, followed by a meaningful interpretation of the different parts of the inequality, and ends with formulating undiscovered properties/knowledge related to the system or process. This approach has been illustrated in Figure 1b.

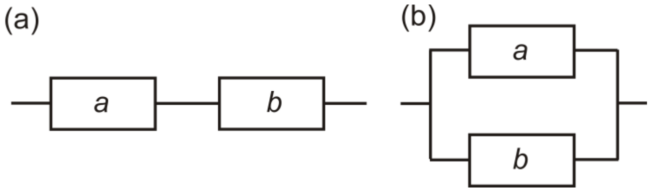

Figure 1. Parts of electrical circuit demonstrating the connection between algebraic inequalities and physical reality; (a) resistors connected in series and (b) resistors connected in parallel.

The inverse approach of the method of algebraic inequalities is rooted in the principle of non-contradiction (Todinov 2020a): *if the variables and the different terms of a correct algebraic inequality can be interpreted as parts of a system or process, in the real world, the system or process exhibit properties or behaviour that are consistent with the prediction of the algebraic inequality.* In short, the realization of the process/experiment yields results that do not contradict the algebraic inequality.

To clarify this principle, consider the correct algebraic inequality

$$a+b \geq 4\frac{1}{1/a+1/b}$$

valid for any positive values a and b.

The left- and right-hand side of this inequality can be interpreted by noticing that if the variables a and b stand for the resistances of two elements, the left-hand side of the inequality can be interpreted as the equivalent resistance of the elements connected in series (Figure 1a). The right-hand side of the inequality can be interpreted as the equivalent resistance of the same elements connected in parallel, multiplied by 4 (Figure 1b). The inequality predicts that *the equivalent resistance of two elements connected in series is more than four times greater than the equivalent resistance of the same elements connected in parallel, irrespective of the individual resistances of the elements*.

If physical measurements of the equivalent resistances of the arrangements in Figure 1a and *b* are conducted, they will only confirm the prediction from the algebraic inequality: that for any combination of values a and b for the resistances of the two elements, the equivalent resistance in series is always more than four times greater than the equivalent resistance of the same elements connected in parallel. Equality is attained only for $a=b$.

The inverse approach related to creating meaningful interpretation for existing non-trivial abstract inequalities and attaching it to a real system or process has only been partially explored (Todinov 2020c) and this determines the topic of the present paper.

The key idea of this paper is that non-trivial algebraic inequalities can be interpreted meaningfully and the new knowledge extracted from the interpretation can be used for optimising systems and processes in diverse areas of science and technology.

The paper demonstrates that the knowledge extracted from the interpretation of non-trivial inequalities is non-trivial and cannot be reached intuitively. In this respect, the paper focuses on the interpretation of inequalities belonging to a general class that permits segmentation or aggregation of controlling factors.

The proposed class of algebraic inequalities is very useful for design in general and its application is demonstrated on systems and processes from diverse application domains. The applications range as follows: (i) developing light-weight designs, (ii) reducing the maximum dynamic force during impact (iii) reducing the stiffness of a mechanical assembly by a cyclic permutation of the elastic elements building the assembly (iv) determining the link between the deflections of elastic elements connected in series and parallel, whose stiffness values are unknown and (v) optimising the allocation of resources to maximise the expected benefit.

All algebraic inequalities presented in the paper have been validated by using a Monte-Carlo simulation algorithm described in (Todinov 2020a).

2. A general class of algebraic inequality providing the opportunity for segmentation of controlling factors

Consider the general inequalities

$$f(a_1+a_2+...+a_n,\ b_1+b_2+...+b_n) \le k[f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n)] \qquad (1)$$

where k is a positive constant and the variables a and b in inequality (1) stand for additive controlling factors.

If the direction of inequality (1) is reversed, the inequality

$$f(a_1+a_2+...+a_n,\ b_1+b_2+...+b_n) \ge k[f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n)] \qquad (2)$$

is obtained.

Inequality (1) provides the opportunity for increasing the effect of the controlling factors $a = a_1+...+a_n$ and $b = b_1+...+b_n$ by segmenting them into smaller parts $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$.

Similarly, inequality (2) provides the opportunity for increasing the effect of the controlling factors a and b by segmenting them into smaller parts $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$.

For a single controlling factor and $k=1$, inequality (1) transform into

$$f(a_1+a_2+...+a_n) \le f(a_1)+f(a_2)+...+f(a_n) \qquad (3)$$

It can also be shown by induction that inequality (3) also follows from the definition

$$f(x_1+x_2) \le f(x_1)+f(x_2) \qquad (4)$$

of a sub-additive function $f(x)$ defined in the Euclidean space of one dimension, for any positive values x_1 and x_2 in the domain of definition (Alsina and Nelsen 2010). Thus, if $f(x)$ is a concave function with domain $[0,\infty]$ and range $[0,\infty]$ then the function $f(x)$ is sub-additive and inequality (4) holds (Todinov 2020c).

For two controlling factors and $k=1$, the general inequality (1) transforms into

$$f(a_1+a_2+...+a_n,\ b_1+b_2+...+b_n) \le f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n) \qquad (5)$$

Similar to Inequality (3), inequality (5) can also be obtained by mathematical induction from the definition of a sub-additive function of two variables:

$$f(x_1+x_2,\ y_1+y_2) \le f(x_1,y_1)+f(x_2,y_2) \qquad (6)$$

In the Euclidean space of two dimensions, the sub-additive function $f(x,y)$ satisfies the inequality (6) for any pair of points (x_1,y_1) and (x_2,y_2).

Properties of multivariable sub-additive functions have already been discussed (Rosenbaum 1950). From the general class of inequalities (2), for a single controlling factor and $k=1$, the inequality

$$f(a_1+a_2+...+a_n) \ge f(a_1)+f(a_2)+...+f(a_n) \quad (7)$$

is obtained. The function $f()$ in inequality (7) is known as *super-additive function of a single variable.* For two controlling factors and $k=1$, the inequality

$$f(a_1+a_2+...+a_n,\, b_1+b_2+...+b_n) \ge f(a_1,b_1)+f(a_2,b_2)+...+f(a_n,b_n) \quad (8)$$

is obtained from the general class of inequalities (2). The function $f()$ in inequality (8) is known as *super-additive function of two variables.*

Inequalities (1),(2),(3),(5),(7) and (8) have a number of powerful potential applications in various domains of science and technology for increasing/decreasing the effect of additive quantities (factors). Inequality (5), for example, effectively states that the effect of the additive quantities $a=\sum_{i=1}^{n} a_i$ and $b=\sum_{i=1}^{n} b_i$ can be increased by segmenting them into smaller parts a_i and b_i, $i=1,...,n$ and accumulating their individual effects $f(a_i,b_i)$, represented by the sum of the terms on the right-hand side of inequality (5).

Inequalities (1)-(8) have a universal application in science and technology. *A relevant inequality from the general classes (1) and (2) can be applied by any user, in any application domain as long as variables a_i, b_i and terms $f(a_i)$, $f(a_i,b_i)$ are additive quantities*. The only requirement in applying the inequalities is the requirement the variables a_i, b_i and the separate terms $f(a_i,b_i)$ to be additive quantities.

Although, the examples used in this paper to illustrate the method based on interpretation of algebraic inequalities are relatively simple, no limitations exist with respect to the size of the system or process.

Examples of additive quantities are mass, weight, amount of substance, number of particles, volume, distance, energy (kinetic energy, gravitational energy, electric energy, elastic energy, surface energy, internal energy), work, power, heat, force, momentum, electric charge, electric current, heat capacity, electric capacity, resistance (when the elements are in series), enthalpy, fluid flow.

In contrast, non-additive quantities usually characterise the object/system locally and do not change with changing the size of the supporting objects/systems. Such properties are 'temperature', 'pressure', 'density', 'concentration', 'hardness', 'velocity', 'surface tension', etc. Inequalities similar to (5) and (8) can be generalised for any number of factors.

3. Light-weight design by interpretation of an algebraic inequality based on a single-variable sub-additive function

Consider the algebraic inequality

$$ca^{2/3} \le ca_1^{2/3} + ca_2^{2/3} + ... + ca_n^{2/3} \quad (9)$$

where *c* is a positive constant and the controlling factor $a=a_1+a_2+...+a_n$ is an additive positive quantity. Inequality (9) is a special case of the general inequality (1) for a single variable and k=1.

The second derivative of the function $y=ca^{2/3}$ is negative ($d^2(ca^{2/3})/da^2 = -(2/9)ca^{-4/3} < 0$) therefore, the function $y=ca^{2/3}$ is concave. Because the power function ca^p where $0 < p < 1$ is a concave function, and the range of $y = ca^p$ is $[0, \infty)$ in the domain $[0, \infty)$ for the argument *a*, inequality (9) follows from the properties of sub-additive functions (Todinov 2020c).

If the variable *a* in inequality (9) is interpreted as magnitude of a load on a cylindrical cantilever beam (Figure 2a), then $ca^{2/3}$ on the left-hand side of the inequality can be interpreted as 'minimum volume of a single cantilever beam necessary to support a force with magnitude *a*'. This interpretation corresponds to the design alternative in Figure 2a.

Indeed, the volume of a cylindrical cantilever beam with radius *r* [m] of the cross section and length *L* [m] is given by

$$V = \pi r^2 L$$

If σ_{cr} is the allowable bending stress, from mechanics of materials (Gere and Timoshenko 1999; Hearn 1985):

$$\sigma_{cr} = \frac{M}{I} r \tag{10}$$

where $M = aL$ is the loading moment and $I = \pi r^4 / 4$ is the second moment of area for a circular cross section. Substituting the second moment of area I in (10) gives

$$\sigma_{cr} = \frac{4aL}{\pi r^3} \tag{11}$$

from which, for the minimum radius of the beam capable of supporting the load with magnitude *a*, $r = \left(\frac{4aL}{\pi \sigma_{cr}} \right)^{1/3}$ is obtained. After substituting *r* in $V = \pi r^2 L$, the volume *V* of the beam with the smallest radius necessary for supporting the loading force *a* becomes

$$V = ca^{2/3} \tag{12}$$

where $c = \frac{4^{2/3} \pi^{1/3} L^{5/3}}{\sigma_{cr}^{2/3}}$.

Supporting the load with magnitude *a*, can also be done by an alternative design option consisting of two cantilever beams with length *L*, loaded with forces with magnitudes $a_1 = a_2 = a / 2$ (Figure 2b).

The two design options with the same function of supporting a total load with magnitude '*a*', depicted in Figure 2a and 2b can now be compared through inequality (9). The comparison shows that the minimum combined volume of the beams necessary to support the loads a_1 and a_2 can be reduced if, instead of two beams, a single cantilever beam with a larger radius is used, loaded with force $a = a_1 + a_2$ equal to the sum of the loading forces $a_1 = a_2 = a / 2$ (Figure 2a).

Indeed, according to inequality (9):

$$ca^{2/3} < c(a/2)^{2/3} + c(a/2)^{2/3} = ca^{2/3} \times (2 / 2^{2/3})$$

Since $(2 / 2^{2/3}) = 1.26$, aggregating the loads and selecting the design option featuring a single cantilever beam with a larger radius, leads to a significant decrease of the minimum volume of material necessary to support the loads.

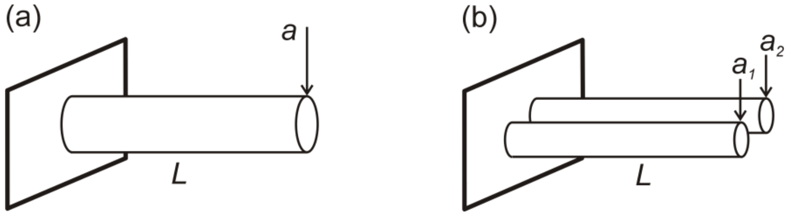

Figure 2 a) A cylindrical cantilever beam with length *L* loaded with a single concentrated force with magnitude *a*; b) two identical cantilever beams with smaller radii, loaded with two forces with magnitudes a_1 and a_2, whose sum is equal to the magnitude *a* of the single force.

Light-weight design developed by using an algebraic inequality will be illustrated by a numerical example. Suppose that the cantilever beam has a length of $L = 2.5$ m, loaded at the end with a force $a = 12$ kN and the material is wood with allowable bending stress $\sigma_{cr} = 15$ MPa.

From $r = \left(\frac{4aL}{\pi\sigma_{cr}}\right)^{1/3} = \left(\frac{4\times 12\times 10^3\times 2.5}{\pi\times 15\times 10^6}\right)^{1/3} = 0.137$, the volume of wood *V* necessary to support a load of magnitude $a = 12$ kN is $V = \pi r^2 L = \pi\times 0.137^2\times 2.5 = 0.147$ m^3.

From $r_1 = r_2 = \left(\frac{4(a/2)L}{\pi\sigma_{cr}}\right)^{1/3} = \left(\frac{4\times 6\times 10^3\times 2.5}{\pi\times 15\times 10^6}\right)^{1/3} = 0.108$, the combined volume V^0 of material of two identical beams needed to support loads of magnitude $a/2 = 6$ kN (the combined load is $a = 12$ kN), is $V^0 = \pi r_1^2 L + \pi r_2^2 L = 2\pi\times 0.108^2\times 2.5 = 0.183$ m^3.

The effect from aggregating the loads is greater for a larger number of aggregated beams. Thus, for a supporting structure consisting of three cantilever beams with length *L*, each loaded with force of magnitude $a_1 = a_2 = a_3 = a/3$, according to inequality (9):

$$ca^{2/3} < c(a/3)^{2/3} + c(a/3)^{2/3} + c(a/3)^{2/3} = ca^{2/3}\times(3/3^{2/3})$$

Since $(3/3^{2/3}) = 1.44$, aggregating the loads into a single load and the three beams into a single beam with a larger radius, leads to an even greater decrease of the mass of the supporting structure.

To summarise, in each case, the same required function (supporting a total load with magnitude *a*), is delivered by two different design options (Figure 2). The two design options however, result in a different required volume of material and the two parts of algebraic inequality (9) represent the output quantity related to the compared design options: 'the minimum volume of material needed for supporting the total load'. Inequality (9) predicts that the single-beam design option is associated with a smaller volume of material necessary to support the load.

For this reason, inequality (9) can be used with success for producing light-weight designs in mechanical and structural engineering.

4. Reducing the maximal dynamic force during impact by interpretation of an algebraic inequality

A special case of the general inequality (2), for *k*=1, is the algebraic inequality

$$\sqrt{ab} \geq \sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n} \tag{13}$$

where both controlling factors $a = a_1 + a_2 + \cdots + a_n$ and $b = b_1 + b_2 + \cdots + b_n$ are additive quantities. The role of the function $f(a,b)$ in inequality (2) is played by the function $f(a,b) \equiv \sqrt{ab}$ in inequality (13).

Inequality (13) can be derived from the Cauchy–Schwarz inequality

$$(x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2 \leq (x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \tag{14}$$

valid for any two sequences of real numbers $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$. Equality in the Cauchy–Schwarz inequality (14) is attained if and only if for any $i \neq j$, $x_i / x_j = y_i / y_j$ are fulfilled (Steele, 2004) which is a well-documented property of the Cauchy–Schwarz inequality.

If the substitutions: $x_1 = \sqrt{a_1}$, $x_2 = \sqrt{a_2}$, ..., $x_n = \sqrt{a_n}$; $y_1 = \sqrt{b_1}$, $y_2 = \sqrt{b_2}$, ..., $y_n = \sqrt{b_n}$ are made in (14), the inequality

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n} \leq \sqrt{\left[\left(\sqrt{a_1}\right)^2 + \cdots + \left(\sqrt{a_n}\right)^2\right] \times \left[\left(\sqrt{b_1}\right)^2 + \cdots + \left(\sqrt{b_n}\right)^2\right]}$$

is obtained, which is effectively inequality (13).

Relevant meaning can now be created for the additive quantities a_i, b_i entering inequality (13). A necessary condition for the interpretation of inequality (13) is that the quantity represented by $\sqrt{a_i b_i}$ must also be an additive quantity.

Consider a body with mass *m* moving along a horizontal trajectory with a speed v and colliding with linear-elastic element with stiffness *c*, acting as a shock absorber (Figure 3a). The maximum dynamic force $F_{\max}$ resulting from the impact, can be determined from equating the kinetic energy $E_k = mv^2 / 2$ of the moving body with the work $A = \frac{F_{\max} \times \delta_{\max}}{2}$ where $\delta_{\max}$ is the spring deflection which corresponds to the force $F_{\max}$ at a deflection $\delta_{\max}$. Since $\delta_{\max} = F_{\max} / c$, the work of the spring force becomes

$$A = \frac{F_{\max} \times \delta_{\max}}{2} = \frac{F_{\max}^2}{2c} \tag{15}$$

Equating kinetic energy and work gives

$$\frac{mv^2}{2} = \frac{F_{\max}^2}{2c} \tag{16}$$

from which, for the maximum dynamic force $F_{\max}$

$$F_{\max} = v\sqrt{cm} \tag{17}$$

is obtained.

For *n* linear springs working in parallel, the equivalent stiffness *c* of the system is an additive quantity because it is a sum of the stiffness values ($c_1, c_2, ..., c_n$) of the individual springs ($c = c_1 + c_2 + ... + c_n$) building the parallel assembly. The mass *m* of the body and the total force $F_{\max}$ are also additive quantities. According to inequality (13)

$$v\sqrt{cm} \geq v\sqrt{c_1 m_1} + v\sqrt{c_2 m_2} + \cdots + v\sqrt{c_n m_n} \tag{18}$$

The left-hand side of inequality (18), can be interpreted as the maximum force $F_{\max}$ for a single mass dampened by the spring assembly with equivalent stiffness *c* (Figure 3a). The right-hand side of inequality (18) can be interpreted as the sum of the dynamic forces

$F_{\max,i} = v\sqrt{c_i m_i}$, resulting from segmenting the mass *m* into smaller masses $m_1, m_2, ..., m_n$ ($m = m_1 + m_2 + ... + m_n$) whose impact is dampened by the individual springs with stiffness values $c_1, c_2, ..., c_n$.

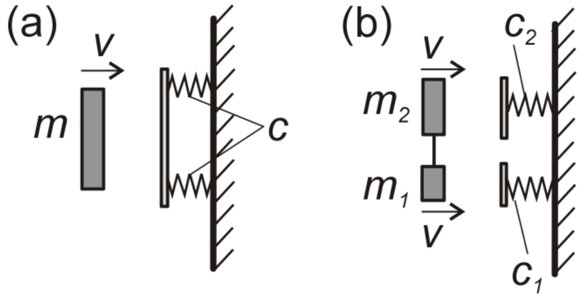

Figure 3 (a) An object with mass *m*, moving with constant velocity v, whose impact is dampened by a spring assembly with equivalent stiffness c b) A pair of objects with masses m_1 and m_2 , ($m = m_1 + m_2$) moving with constant velocity v, whose impact is dampened by the individual springs with stiffness values c_1 and c_2 . ($c = c_1 + c_2$).

Inequality (13) predicts that an impact of a single mass *m* is associated with larger dynamic force compared to the total dynamic force resulting from an impact of the segmented masses dampened by the individual springs. The prediction of inequality (18) can be applied as a basis of a strategy for reducing the dynamic force from an impact between a moving object and a spring assembly.

It may seem that the systems in Figure 3 are two different systems. In fact, Figure 3a and 3b are alternative design options of the same impact-absorbing system built on linear springs. Segmenting the mass and the springs reduces the total dynamic force during an impact.

It needs to be pointed out that *not every segmentation achieves a reduction of the total dynamic force*. Thus, if $\sqrt{m_i} / \sqrt{c_i} = \sqrt{m_j} / \sqrt{c_j}$ for $i \neq j$, which is equivalent to $m_i / c_i = m_j / c_j$, for $i \neq j$, equality is attained in (18). This follows from the properties of the Cauchy-Schwarz inequality (14). In this case, and no decrease of the total dynamic force is present during the impact.

Asymmetry must be present for a decrease in the maximum dynamic force $F_{\max}$ *to occur*. The requirement for asymmetry to reduce the maximum dynamic force $F_{\max}$ is counterintuitive and makes this result difficult to obtain intuitively, bypassing inequality (18).

The knowledge extracted from the interpretation of the non-trivial inequality (18) is non-trivial and cannot be reached intuitively. The conclusions reached about the mechanical systems in Figure (3) cannot be reached intuitively.

5. Reducing the stiffness of a mechanical assembly by cyclic permutation of elastic elements

Inequalities differing from the general inequalities (1) and (2) can also have meaningful interpretation and used to improve the properties of engineering assemblies.

Consider the algebraic inequality

$$\frac{a_1 a_2}{a_1 + a_2} + \frac{a_2 a_3}{a_2 + a_3} + ... + \frac{a_{n-1} a_n}{a_{n-1} + a_n} + \frac{a_n a_1}{a_n + a_1} \le \frac{a_1 + a_2 + ... + a_n}{2} \tag{19}$$

where, $a_i, i = 1,...,n$ are real numbers.

Inequality (19) can be proved by combining the segmentation technique and the well-known arithmetic mean – harmonic mean inequality. Applying the arithmetic mean – harmonic mean inequality gives:

$$\frac{2a_1 a_2}{a_1 + a_2} = \frac{2}{1/a_1 + 1/a_2} \le \frac{a_1 + a_2}{2} \tag{20}$$

$$\frac{2a_2 a_3}{a_2 + a_3} = \frac{2}{1/a_2 + 1/a_3} \le \frac{a_2 + a_3}{2} \tag{21}$$

....

$$\frac{2a_n a_1}{a_n + a_1} = \frac{2}{1/a_n + 1/a_1} \le \frac{a_n + a_1}{2} \tag{22}$$

Adding inequalities (20),...,(22) and dividing both sides by 2 results in inequality (19).
Relevant meaning can be created for the values a_i, b_i, entering inequality (19).

If in the left hand side of (19), in the first term a_2 is replaced by a_1, in the second term a_3 is replaced by a_2 and so on, the equality

$$\frac{a_1^2}{a_1 + a_1} + \frac{a_2^2}{a_2 + a_2} + ... + \frac{a_{n-1}^2}{a_{n-1} + a_{n-1}} + \frac{a_n^2}{a_n + a_n} = \frac{a_1 + a_2 + ... + a_n}{2} \tag{23}$$

is obtained.

If a_i in equality (23) is interpreted as stiffness of an elastic element of type *i*, the ith term on the left hand side of (23) is the equivalent stiffness of two elements of type *i* connected in series (Figure 4a). Hence, the equivalent stiffness of the assembly in Figure 4a is equal to $(a_1 + a_2 + ... + a_n)/2$.

The right-hand side of inequality (19) can then be interpreted as the equivalent stiffness of *n* pairs of elastic assemblies connected in parallel (Figure 4a). Each pair is composed of elastic components of the same type connected in series. The left hand side of inequality (19) can also be interpreted as the equivalent stiffness of *n* pairs of elastic assemblies connected in parallel. The difference from the case in Figure 4a is that now, each pair of elastic elements is composed of elastic elements of different type, connected in series (Figure 4b). In this case, the equivalent stiffness of the assembly is equal to

$\frac{a_1 a_2}{a_1 + a_2} + \frac{a_2 a_3}{a_2 + a_3} + ... + \frac{a_{n-1} a_n}{a_{n-1} + a_n} + \frac{a_n a_1}{a_n + a_1}$, which is the left hand side of inequality (19).

Inequality (19) effectively states that the cyclic permutation of the elastic elements in the pairs composing the assembly in Figure 4a, results in a lower equivalent stiffness of the resultant assembly in Figure 4b.

Inequality (19) will be illustrated by a simple assembly containing 3 pairs of elastic elements only. In Figure (4c), the types of the elastic elements building the separate pairs have been denoted by A,B and C. If the stiffness values of the types A,B and C are a,b and c, correspondingly, inequality (19), becomes

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{a+b+c}{2} \tag{24}$$

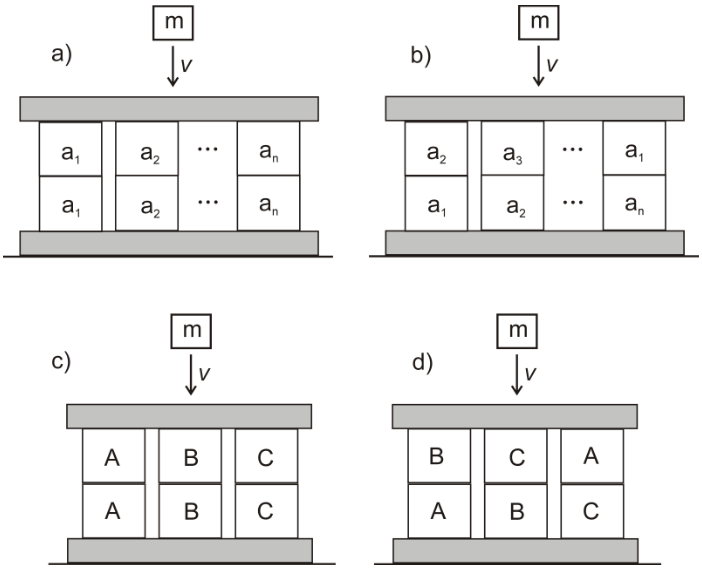

Figure 4. Pairs of elastic elements connected in parallel where the elastic elements in the pairs are connected in series.

For the assembly in Figure (4c) the equivalent stiffness is given by

$$\frac{a^2}{a+a}+\frac{b^2}{b+b}+\frac{c^2}{c+c}=\frac{a+b+c}{2} \tag{25}$$

Hence, the right-hand side of inequality (24) is the equivalent stiffness of the assembly in Figure 4c. The left-hand side of inequality (24) is the equivalent stiffness of the assembly in Figure 4d. Inequality (24) effectively states that the cyclic permutation of elastic elements in the assembly from Figure 4c results in the assembly from Figure 4d, which is characterised by a smaller equivalent stiffness.

The effect of such cyclic permutation can be significant as the next numerical example shows. Suppose that the stiffness values of the elastic elements A,B and C in Figure (4c) are $a=3000$ N/m, $b=9000$ N/m and $c=2000$ N/m, correspondingly. The equivalent stiffness of the assembly in Figure 4c is then

$$\frac{a^2}{a+a}+\frac{b^2}{b+b}+\frac{c^2}{c+c}=\frac{a+b+c}{2}=7000\ \text{N/m}, \tag{26}$$

which is the right hand side of inequality (24). In contrast, the equivalent stiffness of the assembly in Figure 4d is

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}=\frac{3000\times 9000}{(3000+9000)}+\frac{9000\times 2000}{(9000+2000)}+\frac{2000\times 3000}{(2000+3000)}=5086.4\ \text{N/m} \tag{27}$$

which is significantly smaller than the equivalent stiffness of the assembly in Figure (4c). According to the previous section, if a body with mass m strikes the elastic assembly with velocity v, for the maximum force $F_{\max}$ of the impact we have

$$F_{\max}=v\sqrt{cm} \tag{28}$$

where c is the equivalent stiffness of the assembly. Reducing the equivalent stiffness c reduces the maximum dynamic force upon impact. Consequently, a cyclic permutation of the elastic elements in the assembly reduces the maximum dynamic force upon impact.

6. Generating knowledge and optimisation of processes from an interpretation of a new algebraic inequality

Consider the algebraic inequality

$$\frac{a_1}{x_1}+\frac{a_2}{x_2}+...+\frac{a_n}{x_n}\geq n\frac{a_1+a_2+...+a_n}{x_1+x_2+...+x_n} \tag{29}$$

where $a_1 \geq a_2 \geq ... \geq a_n$ and $x_1 \leq x_2 \leq ... \leq x_n$ are positive additive quantities. Inequality (29) is a special case of the general inequality (1), where $k = 1/n$.

The role of the function $f(a,b)$ in inequality (1) is played by the function $f(a,x) = a/x$ in inequality (29).

Inequality (29) can be proved by applying the Chebyshev's inequality, the AM-GM (Arithmetic mean – Geometric mean) inequality and the technique 'strengthening of an inequality'. The classical Chebyshev's sum inequality (Steele 2004) states that for the sequences of real numbers $a_1 \geq a_2 \geq, \ldots, \geq a_n$ and $b_1 \geq b_2 \geq, \ldots, \geq b_n$, the following inequality holds:

$$n(a_1b_1+a_2b_2+\cdots+a_nb_n)\geq(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)$$

Let the setting $b_i = 1/x_i$ be made in the last inequality. Because $x_1 \leq x_2 \leq ... \leq x_n$, from which $b_1 = 1/x_1 \geq b_2 = 1/x_2 \geq ... \geq b_n = 1/x_n$ follows, the conditions for the Chebyshev's sum inequality are fulfilled and the result is the inequality:

$$n\left(\frac{a_1}{x_1}+\frac{a_2}{x_2}+...+\frac{a_n}{x_n}\right)\geq(a_1+a_2+...+a_n)\times(1/x_1+1/x_2+...+1/x_n) \tag{30}$$

For any sequence $b_1, b_2, \cdots, b_n$ of real numbers, the AM-GM inequality states that, the inequality

$$\frac{b_1+b_2+\cdots+b_n}{n}\geq\sqrt[n]{b_1b_2\ldots b_n}$$

always holds. If the setting $b_i = 1/x_i$ is made in the AM-GM inequality, the inequality

$$\frac{1/x_1+1/x_2+...+1/x_n}{n}\geq\frac{1}{\sqrt[n]{x_1x_2...x_n}} \tag{31}$$

is obtained. Substituting $\frac{n}{\sqrt[n]{x_1x_2...x_n}}$, instead of $1/x_1+1/x_2+...+1/x_n$, in the right-hand side of (30), results in the inequality:

$$n\left(\frac{a_1}{x_1}+\frac{a_2}{x_2}+...+\frac{a_n}{x_n}\right)\geq\frac{n(a_1+a_2+...+a_n)}{\sqrt[n]{x_1x_2...x_n}} \tag{32}$$

From the AM-GM inequality,

$$\sqrt[n]{x_1x_2...x_n}\leq\frac{x_1+x_2+...+x_n}{n} \tag{33}$$

holds, and substituting $\frac{x_1+x_2+...+x_n}{n}$ in (32) instead of $\sqrt[n]{x_1x_2...x_n}$, only strengthens inequality (32). The result is the inequality

$$n\left(\frac{a_1}{x_1}+\frac{a_2}{x_2}+...+\frac{a_n}{x_n}\right)\geq\frac{n^2(a_1+a_2+...+a_n)}{x_1+x_2+...+x_n} \quad (34)$$

which, after the division of both sides by *n*, gives inequality (29).

Presence of asymmetry is vital for inequality (29) to hold. For equal ratios $a_i/x_i=a_j/x_j=r$, $i=1,...,n;\ j=1,...,n$, equality is attained in (29).

Indeed, in this case, the left-hand side of (29) becomes $\frac{a_1}{x_1}+\frac{a_2}{x_2}+...+\frac{a_n}{x_n}=nr$ and since $a_i=rx_i$, the right hand side of (29) also gives the same value: $n\frac{a_1+a_2+...+a_n}{x_1+x_2+...+x_n}=n\frac{r(x_1+x_2+...+x_n)}{x_1+x_2+...+x_n}=nr$.

Inequality (29) provides a mechanism for increasing at least *n* times the effect of the aggregated additive quantities $a=a_1+a_2+...+a_n$ and $x=x_1+x_2+...+x_n$ by segmenting them into smaller parts a_i, x_i, $i=1,...,n$ and accumulating their individual effects a_i/x_i. As it stands, in order to apply inequality (29), the ratio a_i/x_i of the additive quantities a_i and x_i must also be an additive quantity.

Even if the ratio a/x is a non-additive quantity, inequality (29), can still be applied if the multiplication by a particular positive factor λ turns the non-additive quantity a/x into an additive quantity $\lambda a/x$. In this case, inequality (29) becomes

$$\left(\frac{\lambda a_1}{x_1}+\frac{\lambda a_2}{x_2}+...+\frac{\lambda a_n}{x_n}\right)\geq\frac{n\lambda(a_1+a_2+...+a_n)}{x_1+x_2+...+x_n} \quad (35)$$

If the variables a_i in inequality (29) are interpreted as loads on *n* elastic elements and x_i are interpreted as the stiffness values of these elastic elements, the terms a_i/x_i can be interpreted as deflections of the elastic elements under these loads. The loads a_i are additive quantities and the stiffness values for elastic elements connected in parallel are also additive quantities.

Suppose also that $a_1=a_2=...=a_n=F/n$. If x_i are ordered in ascending order ($x_1\leq x_2\leq...\leq x_n$), because $a_1=a_2=...=a_n=F/n$, the conditions $a_1\geq a_2\geq...\geq a_n$, $x_1\leq x_2\leq...\leq x_n$ for the validity of inequality (29) are fulfilled and inequality (29) holds.

Inequality (29) can then be re-written as

$$\frac{F/n}{x_1}+\frac{F/n}{x_2}+...+\frac{F/n}{x_n}\geq n\frac{F/n+F/n+...+F/n}{x_1+x_2+...+x_n} \quad (36)$$

which is equivalent to

$$\frac{1}{n^2}\left(\frac{F}{x_1}+\frac{F}{x_2}+...+\frac{F}{x_n}\right)=\frac{F}{x_1+x_2+...+x_n} \quad (37)$$

The right-hand side part of inequality (37) is the deflection of an assembly of *n* elastic elements connected in parallel (see Figure 5a) and loaded with a force with magnitude F. The left-hand side of (37) is the total deflection divided by n^2, of the same *n* elastic elements connected in series and loaded by a single force with the same magnitude F (Figure 5b).

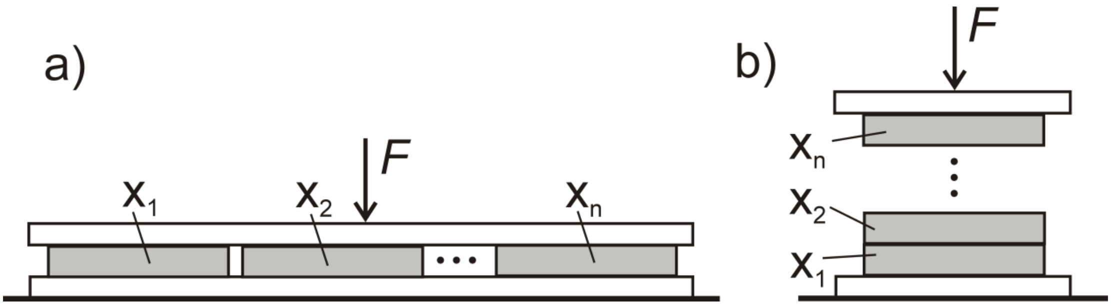

Figure 5. Elastic elements with stiffness values x_i arranged in (a) parallel and (b) series. The series arrangement (b) is loaded with the same force *F* as the parallel arrangement.

The interpretation of inequality (37) states that the deflection of elastic elements connected in parallel is at least n^2 times smaller than the deflection of the same elastic elements connected in series and loaded with a force with the same magnitude, irrespective of the stiffness values of the individual elements.

Inequality (37) can be applied to assess the magnitude of the deflection of assemblies of elastic elements due to switching from parallel to series connection.

Inequality (35) also admits powerful alternative interpretations that can be used for optimising processes. Because of the coefficient of proportionality $\lambda > 0$, the terms $\lambda a_i / x_i$ are additive and inequality (35) has a natural interpretation in many application domains. Suppose that the effect *E* from a particular process is an additive quantity which can be presented as

$$E = \lambda \frac{a}{x} \tag{38}$$

where *a* and *x* are factors controlling the process that are also additive quantities. In equation (38), λ is a constant coefficient, independent of the values of the factors *a* and *x*. In addition, if the factor *a* is segmented into *n* segments a_i, $i = 1,...,n$ and the factor *x* is also segmented into *n* segments x_i, $i = 1,...,n$, the terms $E_i = \lambda \frac{a_i}{x_i}$ are also additive quantities.

According to inequality (35), segmentation of the controlling factors *a* and *x* into smaller segments, will increase at least *n* times the cumulative effect $\sum_{i=1}^{n} E_i$ compared to the effect *E* obtained without segmentation. This significant increase can be demonstrated by the next numerical example involving two segments only.

Suppose that a certain amount *a* of resources (for example, $a = 9$ research units) are available and a particular allocated field which includes *x* different topics (for example, $x = 6$) must be explored experimentally. From past experience, it is known that the expected number of results *E* obtained from the experimentation is proportional to the density a / x of the resources engaged in the selected set of topics, but does not depend on the number of explored topics: $E = \lambda \frac{a}{x}$. The numerical value of the constant λ of proportionality is unknown.

This statement of the optimisation problem fits in the description of the model presented with equation (38). In the absence of segmentation, the expected number of results is $E = \lambda a / x = \lambda \times 9 / 6 = 1.5\lambda$, where λ is unknown coefficient of proportionality.

Suppose that the available resources are segmented into $a_1 = 6$, $a_2 = 3$ and the available topics to be explored experimentally into $x_1 = 2$ and $x_2 = 4$. The first resource segment is paired with the first topics segment and the second resource segment is paired with the second topics segment in such a way that the inequalities $a_1 > a_2$ and $x_1 < x_2$ are fulfilled. In this case, inequality (35) holds and predicts at least a two-fold increase in the expected number of results ($n = 2$). Indeed,

$$E_1 + E_2 = \left(\frac{\lambda a_1}{x_1} + \frac{\lambda a_2}{x_2} \right) = \lambda \times (6/2 + 3/4) = 3.75\lambda$$

From inequality (35):

$$E_1 + E_2 = \left(\frac{\lambda a_1}{x_1} + \frac{\lambda a_2}{x_2} \right) = 3.75\lambda > 2 \times E = 2 \times \lambda \frac{a}{x} = 3.0\lambda$$

In inequality (35), the constant λ is cancelled and its specific numerical value does not affect the predictions of the inequality regarding the effect of segmentation.

Here, it is important to note that not every segmentation achieves more than a two-fold increase in the expected number of results. To achieve such an increase, the conditions of inequality (35) must be fulfilled. For example, more than a two-fold increase in the expected results is not present if the available resources are segmented into $a_1 = 6$, $a_2 = 3$ and the available topics to be explored experimentally are segmented into $x_1 = 4$ and $x_2 = 2$ topics. Indeed, in this case, $E = \lambda a / x = \lambda \times 9/6 = 1.5\lambda$ and

$$E_1 + E_2 = \left(\frac{\lambda a_1}{x_1} + \frac{\lambda a_2}{x_2} \right) = \lambda \times (6/4 + 3/2) = 3.0\lambda = 2 \times (1.5)\lambda$$

To summarise, the knowledge extracted from the interpretation of the non-trivial inequalities (29) and (35) is non-trivial and the conclusions reached about the mechanical system cannot be reached intuitively. Indeed, the interpretation of the abstract inequality (29) helped find an overlooked fundamental property of elastic components which has never been reported in the mature fields of mechanical engineering and stress analysis. No such result has been reported in modern comprehensive textbooks in stress analysis and mechanical engineering (Budynas 1999; Gere and Timoshenko 1999; Hearn 1985; Collins 2003; Norton 2006; Childs 2014; Budynas and Nisbett 2015; Pahl et al. 2007; Mott et al. 2018; French 1999; Gullo and Dixon) which indicates that lack of knowledge of the method of algebraic inequalities made a fundamental property of elastic elements connected in series and parallel invisible to domain experts.

By interpreting algebraic inequalities, new results are always obtained, without the need of any forward analysis. The generated new knowledge can then be used for optimising variety of systems and processes in any area of science and technology.

CONCLUSIONS

1. A special class of general inequalities has been identified that provides the opportunity for a segmentation an aggregation of controlling factors. It is demonstrated that inequalities belonging to this class can always be interpreted meaningfully if the variables and separate terms of the inequalities represent additive quantities. The generated new knowledge can then be used for optimising systems and processes in diverse areas of science and technology.

2. The interpretation of a new inequality from the proposed general class admits interpretations in various domains that can be used for generating for generating new knowledge and optimising processes. The inequality has been applied for establishing that the deflection of n elastic elements connected in series is at least n^2 times larger than the deflection of the same elements connected in parallel, irrespective of the individual stiffness values of the elements.

3. The paper demonstrates that even if the separate terms of the new inequality are not additive, in some cases they can be transformed from non-additive to additive by a multiplying both sides of the inequality with appropriate constant. This technique has been used to establish a strategy for optimising the allocation of resources in order to maximise the expected benefit.

4. A method for optimising the design of systems and processes has been introduced that consists of interpreting the left- and the right-hand side of a correct algebraic inequality as outputs of two alternative design configurations delivering the same required function. In this way, the superiority of one of the configurations is established immediately. The proposed method opens wide opportunities for enhancing the performance of systems and processes from diverse application domains.

5. The meaningful interpretation of an algebraic inequality based on a single-variable sub-additive function led to developing a light-weight design for a supporting structure based on cantilever beams.

6. The meaningful interpretation of a new algebraic inequality based on the proposed general class of inequalities led to a method for reducing the maximum dynamic force during an impact with a linear-elastic element.

7. The interpretation of a new algebraic inequality based on a multivariable super-additive function led to a method for decreasing the stiffness of a mechanical assembly by a cyclic permutation of the elastic elements building the assembly.

8. The developments presented in this paper can be expanded significantly by interpreting the general inequalities in the context of various specific domains: electric engineering, manufacturing, all areas of physics, economics and operational research. The proposed general class of algebraic inequalities is very promising. The inequalities belong to this class can be easily interpreted meaningfully if their variables and terms represent additive quantities.

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