RELIABILITY-RELATED INTEPRETATIONS OF ALGEBRAIC INEQUALITIES

Michael T. Todinov
School of Engineering, Computing and Mathematics
Oxford Brookes University,
Oxford, Wheatley, OX33 1HX,
mtodinov@brookes.ac.uk

Abstract

New results related to maximising the reliability of common systems with interchangeable redundancies at a component level have been obtained by using the method of algebraic inequalities. It is shown that for systems with independently working components with interchangeable redundancies, the system reliability corresponding to a symmetric arrangement of the redundant components is always inferior to the system reliability corresponding to an asymmetric arrangement of the redundant components, irrespective of the probabilities of failure of the different types of components.

It is also shown that for series-parallel systems, the system reliability is maximised by arranging the main components in ascending order of their probabilities of failure while the redundant components are arranged in descending order of their probabilities of failure.

Finally, the paper derives rigorously the highly counter-intuitive result that if two components must be selected from *n* batches containing reliable and faulty components with unknown proportions, the likelihood that both components will be reliable is maximised by selecting both components from a randomly selected batch.

Keywords: method of algebraic inequalities; maximising system reliability; improving reliability; reducing risk; generic methods for reliability improvement

1. Introduction

Algebraic inequalities have numerous applications in mathematics and engineering and discussion related to algebraic inequalities can be found in (Kazarinoff 1961; Bechenbach and Bellman 1961; Engel 1998; Hardy et al. 1999; Fink 2000; Pashpatte 2005; Marshall et al. 2010; Steele 2004; Sedrakyan and Sedrakyan 2010; Cvetkovski 2012; Saif, 2007).

In engineering (Cloud et al. 2014; Samuel and Weir 1999; Rastegin 2012; Liu and Lin, 2013), design constraints are often expressed through algebraic inequalities. The design constraints form a design space for the design parameters which guarantees the absence of failure modes.

In reliability and risk research, inequalities have been used as a tool for characterisation of reliability functions (Ebeling, 1997; Xie and Lai, 1998; Makri and Psillakis, 1996; Hill et al., 2013; Berg and Kesten, 1985; Kundu and Ghosh, 2017; Dohmen, 2006).

In reliability theory, inequalities have also been used to study the preservation properties related to coherent systems (Navaro et al., 2016).

Previous publications on the application of algebraic inequalities in engineering (Cloud et al 2014) are primarily focused on generating upper and lower bounds.

An application of the method of algebraic inequalities to improve reliability and reduce risk has been demonstrated in (Todinov, 2020) where algebraic inequalities have been used to rank the reliabilities of systems with unknown reliabilities of their components. This approach can be summarised as follows. For each of the two competing alternatives 1 and 2 of a system, a reliability network is built first. Next, by using methods from system reliability analysis, the system reliabilities R_1 and R_2 of the competing alternatives are determined. The final step is trying to prove one of the inequalities $R_1 - R_2 > 0$ or $R_2 - R_1 > 0$ (irrespective of the specific reliabilities of the components), which demonstrates the superior reliability of one of the alternatives. We need to point out here that the direct approach of the method of algebraic inequalities has a limitation: is not guaranteed to work for all compared alternatives. This means that in some cases it is not possible to prove any of the of the inequalities $R_1 - R_2 > 0$ or $R_2 - R_1 > 0$, irrespective of the reliabilities of the components building the systems.

Very few publications exist that are related to generating new knowledge by interpreting non-trivial algebraic inequalities, which is subsequently used for optimising engineering systems or processes. The inverse approach in using the method of algebraic inequalities, is based on a recently formulated by the author principle of non-contradiction: *if the variables and the different terms of a correct algebraic inequality can be interpreted as parts of a system or process, in the physical world, the system or process exhibit properties or behaviour that are consistent with the prediction of the algebraic inequality.*

Consider a particular process/system that can be developed in two different variants. Suppose that the left and right part of a correct algebraic inequality can be interpreted as models of the outputs related to the competing variants. The algebraic inequality can then be used to establish which of the competing alternatives is superior.

Algebraic inequalities are suitable for modelling processes and systems with inherent unstructured uncertainty, which is a significant advantage. This is because algebraic inequalities do not require values of the variables present in the inequalities. This aspect makes algebraic inequalities superior to conventional approaches for modelling uncertainty which require various assumptions.

Modelling based on probabilities and Monte Carlo simulations for example, requires assumptions related to probabilistic models assigned to the random variables. These assumptions often do not reflect correctly the modelled phenomena and lead to incorrect predictions.

Furthermore, the method of algebraic inequalities is a domain-independent method. It can be applied for reliability improvement across very different domains of human activity. The method of algebraic inequalities: (i) does not rely on reliability data; (ii) is appropriate for new designs, with no failure history and (iii) encourages simple, low-cost solutions.

The physical interpretation of algebraic inequalities involves meaningful physical interpretation of the variables present in the inequalities and meaningful physical interpretation of the separate terms of the inequality. Following this approach, new knowledge can be derived for the reliability of systems in series, with arbitrary number of independently working components and interchangeable redundancies. This knowledge can serve as a basis for increasing the reliability of common systems.

The question on optimal redundancy allocation for systems with series arrangement of independently working components has been considered in (Valdes and Zequeira, 2006). The results obtained however, have limited validity because they are related to very simple systems, incorporating only two-components.

For other inequalities obtained from the standard Muirhead's inequality, important meaningful interpretations have also been made. Thus, for suppliers with unknown fractions of reliable products a powerful strategy has been derived for maximising the probability that all purchased components will be reliable.

2. Improving the reliability of a system with interchangeable redundancies by interpreting algebraic inequalities

If the variables $a_1, a_2, ..., a_n$ entering a correct algebraic inequality are subjected to the constraints $0 \le a_1, a_2, ..., a_n \le 1$, $i = 1, ..., n$, they can, for example, be interpreted as probabilities of failure of components working independently from one another. In addition, if the left or right-hand side of an algebraic inequality is composed of products of terms of the type: $(1 - a_i a_j)$, the term $(1 - a_i a_j)$ can be interpreted as the reliability of a section including two components logically arranged in parallel (Figure 1a). The product $(1 - a_i a_j)(1 - a_{i+1} a_{j+1})...(1 - a_r a_s)$ of several such terms can be interpreted as the reliability of a series-parallel system including a number of sections logically arranged in series, within each of which, the components are logically arranged in parallel (Figure 1b).

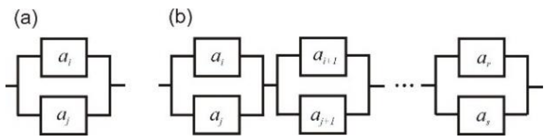

Figure 1. Series-parallel systems, including sections whose reliabilities are given by products of terms $(1 - a_i a_j)$.

Accordingly, the left and right-hand side of inequalities including these terms can be interpreted as reliabilities of alternative system configurations. Next, through algebraic inequalities, the intrinsic reliabilities of the alternative configurations can be compared.
To illustrate this approach, consider the simple algebraic inequality:

$$(1 - a_1^2)(1 - a_2^2) \le (1 - a_1 a_2)(1 - a_2 a_1) \quad (1)$$

where $0 \le a_1 \le 1$; $0 \le a_2 \le 1$. Proving this inequality is equivalent to proving the equivalent inequality

$$1 - a_1^2 - a_2^2 + a_1^2 a_2^2 \le 1 - 2a_1 a_2 + a_1^2 a_2^2 \quad (2)$$

which, in turn can be proved by proving the equivalent inequality

$$a_1^2 + a_2^2 - 2a_1 a_2 \ge 0 \quad (3)$$

Inequality (3) however, is true because $a_1^2 + a_2^2 - 2a_1 a_2 = (a_1 - a_2)^2$ is non-negative. Inequality (1) can be generalised for more than two types of components. Thus, for $n \ge 2$ types of components $A_1, A_2, ..., A_n$ with probabilities of failure $a_1, a_2, ..., a_n$, correspondingly, inequality (1) is generalised to

$$\begin{aligned}&(1 - a_1^2)(1 - a_2^2)...(1 - a_n^2) \le \\ &(1 - a_1 a_2)(1 - a_2 a_3)...(1 - a_n a_1)\end{aligned} \quad (4)$$

Proof. Inequality (4) can be proved by induction. For $n = 2$, inequality (4) coincides with inequality (1) which has been shown to be true.
Without loss of generality, we can assume that either $a_1 \le a_2 \le ... \le a_k \le a_{k+1}$ or $a_1 \ge a_2 \ge ... \ge a_k \ge a_{k+1}$ holds. (The probabilities of failure a_i of components can always be arranged in ascending or descending order).
Let us assume that inequality (4) is true for $n = k$ (induction hypothesis):

$$\begin{aligned}&(1 - a_1^2)(1 - a_2^2)...(1 - a_k^2) \le \\ &(1 - a_1 a_2)(1 - a_2 a_3)...(1 - a_k a_1)\end{aligned} \quad (5)$$

It can be shown that the inequality is also true for $n = k + 1$.
Multiplying both sides of inequality (5) by $(1 - a_{k+1}^2)$ gives the inequality

$$\begin{aligned}&(1 - a_1^2)(1 - a_2^2)...(1 - a_k^2)(1 - a_{k+1}^2) \le \\ &(1 - a_1 a_2)(1 - a_2 a_3)(1 - a_k a_1)(1 - a_{k+1}^2)\end{aligned} \quad (6)$$

If it can be shown that

$$\begin{aligned}&(1 - a_k a_1)(1 - a_{k+1}^2) \le \\ &(1 - a_k a_{k+1})(1 - a_{k+1} a_1)\end{aligned} \quad (7)$$

This means that replacing the expression $(1 - a_k a_1)(1 - a_{k+1}^2)$ in the right-hand side of inequality (6) by the larger expression

$(1-a_k a_{k+1})(1-a_{k+1}a_1)$, will only strengthen inequality (6).

Consequently, to prove inequality (7), the equivalent inequality:

$$\begin{aligned} &1-a_{k+1}^2 - a_k a_1 + a_1 a_k a_{k+1}^2 \le \\ &1-a_{k+1}a_1 - a_k a_{k+1} + a_1 a_k a_{k+1}^2 \end{aligned} \tag{8}$$

must be proved, which is obtained from expanding the left- and right-hand side of (7). Proving inequality (8) is equivalent to proving

$$a_{k+1}^2 + a_k a_1 - a_{k+1}a_1 - a_k a_{k+1} \ge 0 \tag{9}$$

The left-hand side of (9) can be factorised as:

$$\begin{aligned} &a_{k+1}^2 + a_k a_1 - a_{k+1}a_1 - a_k a_{k+1} = \\ &= (a_{k+1}-a_1)(a_{k+1}-a_k) \end{aligned} \tag{10}$$

and because the probabilities of failure are arranged in ascending $a_1 \le a_2 \le ... \le a_k \le a_{k+1}$ or descending ($a_1 \ge a_2 \ge ... \ge a_k \ge a_{k+1}$) order, the inequality $(a_{k+1}-a_1)(a_{k+1}-a_k) \ge 0$ holds. The case $n = k+1$ has been proved. Because inequality (4) is true for $n = 2$, the inequality is also true for $n = 3$, $n = 4$ and for any other $n \ge 2$.

A natural interpretation of inequality (4) can now be given in terms of reliability of a series-parallel system including components that work and fail *independently* from one another. If the variables a_i in inequality (4) are interpreted as probabilities of failure of statistically independent components A_i, the left-hand side of inequality (4) gives the reliability of the system configuration in Figure 2a while the right-hand side of inequality (4) gives the reliability of the system configuration in Figure 2b. Figure 2a and 2b depict reliability networks of common systems with interchangeable active redundancies at a component level.

Suppose that components A_i ($i = 1,...,n$) stand for interchangeable sensors of n different types logically arranged in series. The sensors collect critical information from n zones in the system. For the system to operate successfully, at least a single sensor from each zone (block) must be operational. Each zone (block) includes a pair of sensors working in parallel.

Any particular type of sensor can work as a redundant sensor in any zone (block).

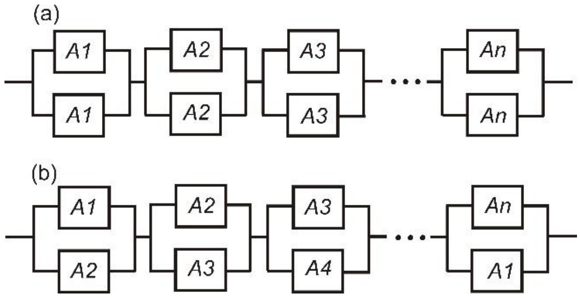

Figure 2. Reliability network of two alternative systems built with the same types and number of components A1,A2,...,An.

The interpretation of inequality (4) yields new knowledge: For systems built with components that work and fail independently from one another, the reliability of the system with asymmetrical arrangement of the active redundancies in Figure 2b is always greater than the reliability of the system in Figure 2a with symmetrical arrangement of the redundancies. This result holds *irrespective of the actual reliabilities (or probabilities of failure) of the components building the systems*. A conclusion has been reached that the natural arrangement of the same type redundancies (A1/A1, A2/A2,...,An/An) results in a smaller system reliability compared to an asymmetrical arrangement A1/A2, A2/A3,...,An/A1.

The same approach to improving system reliability is valid for other interchangeable components, for example, for interchangeable switches of different types, seals, pumps, etc.

The knowledge derived from the interpretation of inequality (4) can be used for optimising series-parallel systems. The system reliability is substantially increased if the symmetry in the arrangement of the different types of redundant components is destroyed.

If the probabilities of failure a_i of the components are known, the reliability of the system in Figure 3a, with symmetrical arrangement of the redundant components, can be maximised by arranging the components in the upper branches in ascending order of their probabilities of failure while the components in the lower branches are arranged in descending order of their probabilities of failure. The reliability network of the system with the largest reliability is given in Figulre 3b.

Without loss of generality, we can always assume that the components A1, A2,...,An in the upper branches in Figure 3b are arranged according to their probabilities of failure in ascending order. Suppose that the main components (in the upper branches of Figure 3b) are arranged in ascending order of their probabilities of failure: $a_1 \le a_2 \le ... \le a_n$. The redundant components (the lower branches of Figure 3b) are arranged in descending order of their probabilities of failure: $a_n \ge a_{n-1} \ge ... \ge a_1$.

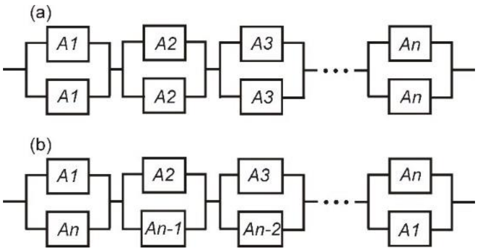

Figure 3. a) Reliability network with interchangeable redundancies b) Reliability network of the system characterised by the largest reliability.

It can then be shown that the permutation in Figure 3b is characterised by the largest reliability, compared to the any other permutation. Since the reliability of the system in figure 3b is given by $R = (1 - a_1 a_n)(1 - a_2 a_{n-1})...(1 - a_n a_1)$, it is effectively required to show that the inequality

$$(1 - a_1 a_n)(1 - a_2 a_{n-1})...(1 - a_n a_1) \ge$$
$$(1 - a_1 a_{p1})(1 - a_2 a_{p2})...(1 - a_n a_{pn}) \quad (11)$$
$$0 \le (1 - a_i a_j) \le 1$$

holds, where $a_{p1}, a_{p2}, ..., a_{pn}$ is any particular permutation of the components (probabilities of failure) in the lower branches.

Proof. Inequality (11) can be proved by using *the extreme principle*. Suppose that there is an arrangement where the components in the lower branches are not arranged in descending order of their probabilities of failure and the system reliability given by the product $(1 - a_1 a_{p1})(1 - a_2 a_{p2})...(1 - a_n a_{pn})$ is the largest possible. In this case, there must exist at least two terms $(1 - a_i a_{px})(1 - a_j a_{py})$ where $i < j$; $a_i < a_j$; and $a_{px} < a_{py}$. Otherwise, if no such terms can be found, for which $a_{px} < a_{py}$, the components in the lower branches would have been already arranged in descending order.

We will show that if $a_{px} < a_{py}$, the system reliability given by the product $(1 - a_1 a_{p1})(1 - a_2 a_{p2})...(1 - a_n a_{pn})$ cannot be the largest possible which leads to a contradiction with the assumption that this is the largest possible system reliability.

Compare the product $(1 - a_i a_{px})(1 - a_j a_{py})$ with the product $(1 - a_i a_{py})(1 - a_j a_{px})$ obtained by swapping the redundant components with indices $'px'$ and $'py'$ in the lower branches. We will show that

$$(1 - a_i a_{px})(1 - a_j a_{py}) < (1 - a_i a_{py})(1 - a_j a_{px}) \quad (12)$$

Expanding the left- and right-hand side of inequality (12) leads to the equivalent inequality

$$-a_i a_{px} - a_j a_{py} < -a_i a_{py} - a_j a_{px} \quad (13)$$

Inequality (13) is equivalent to the inequality

$$(a_{py} - a_{px})(a_j - a_i) > 0 \quad (14)$$

Inequality (14) is true because $a_j > a_i$ and $a_{py} > a_{px}$.

This shows that inequality (12) holds and, contrary to the assumption, the system reliability is not the largest possible.

This means that a larger product $(1 - a_1 a_{p1})(1 - a_2 a_{p2})...(1 - a_n a_{pn})$

cannot be obtained from a permutation $a_{p1}, a_{p2}, ..., a_{pn}$ that is different from a permutation corresponding to arranging the values a_{pi} in descending order: $a_{p1} \geq a_{p2} \geq, ..., \geq a_{pn}$.
which completes the proof of inequality (11).

It is difficult to see how these general results related to the reliability of a series-parallel system with *n* sections could be obtained without using the non-trivial inequalities (4) and (11).

The difference in the system reliabilities of the competing systems in Figure 2a,2b and Figure 3a,3b, for example, can be very large as the next, deliberately simplified example based on n=4 components demonstrates. Thus, for interchangeable sensors of types *A*,*B*,*C* and *D*, characterised by probabilities of failure of $a_1 = 0.28$, $a_2 = 0.53$, $a_3 = 0.82$ and $a_4 = 0.85$) for two years of continuous operation, the arrangement in Figure 2a ($n = 4$) is characterised by a system reliability of

$$R_a = (1-0.28^2)(1-0.53^2) \times (1-0.82^2)(1-0.85^2) = 0.06$$

while the arrangement in Figure 2b is characterised by a system reliability of

$$R_b = (1-0.28\times 0.53)(1-0.53\times 0.82) \times (1-0.82\times 0.85)(1-0.85\times 0.28) = 0.11$$

The largest system reliability is obtained for the arrangement in Figure 3b ($n = 4$):

$$R_b = (1-0.28\times 0.85)(1-0.53\times 0.82) \times (1-0.82\times 0.53)(1-0.85\times 0.28) = 0.186$$

The result related to maximising system reliability can, for example, be used for any system involving a new (N), medium-age (M) and old component (O) and interchangeable redundancies of the corresponding age (Figure 4a). Making the natural assumption $a_N < a_M < a_O$ for the probabilities of failure of the components, arranging the components and the redundancies as is shown in Figure 4b yields the largest system reliability. This arrangement always brings the largest system reliability, irrespective of the specific probabilities of failure characterising the components. The only requirement is the ranking assumption $a_N < a_M < a_O$ related to the probabilities of failure which, after eliminating early-life failures, commonly holds for new, medium-age and old components.

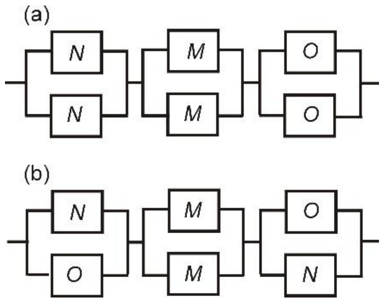

Figure 4. a) Reliability network with interchangeable redundancies involving new, medium-age and old components b) The reliability network of the system characterised by the largest reliability.

An inequality similar to inequality (4) can also be proved for a more complex system, for example, for the system in Figure 5a.

It can be shown that the inequality

$$(1-a_1^m)(1-a_2^m)...(1-a_n^m) \leq (1-a_1^{m-t}a_2^t)(1-a_2^{m-t}a_3^t)...(1-a_n^{m-t}a_1^t) \quad (15)$$

holds, where $m > 2$ and $1 \leq t \leq m$.

As a result, the system in Figure 5a, has a lower reliability compared to the system with asymmetric redundancy arrangement (corresponding to $t = 1$) in Figure 5b.
In this case, the inequality

$$(1-a_1^m)(1-a_2^m)...(1-a_n^m) \leq (1-a_1^{m-1}a_2)(1-a_2^{m-1}a_3)...(1-a_n^{m-1}a_1) \quad (16)$$

is obtained from (15), which corresponds to $t = 1$. The proof of inequality (15) is similar to the proof presented for inequality (4) and has been given in the Appendix.

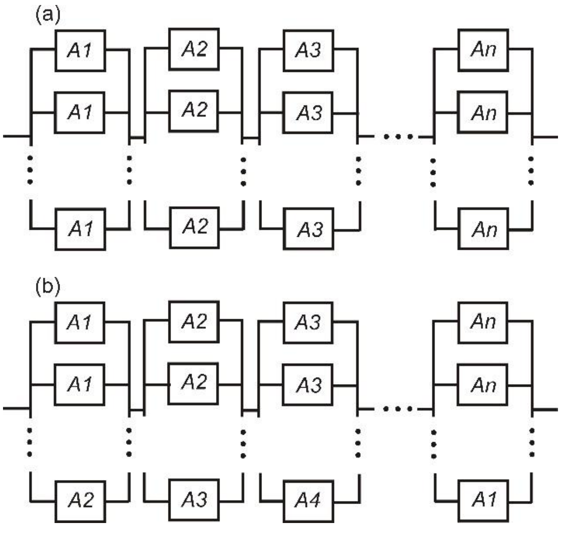

Figure 5. Reliability networks of two alternative systems: a) a system with m-1 active redundant components and b) a system for which the redundant components in one of the branches have been cyclically shifted.

3. Interpretation of an algebraic inequality to increase the probability of selecting a set of reliable components

Commonly, the components sourced from *n* batches (suppliers) can be of two varieties: 'reliable components' and 'faulty components'. Two components (e.g. ball bearings) must be selected from the *n* batches and installed in a system. For the system to work properly, *both selected components must be reliable*.

Suppose that the percentage of reliable components characterising the *n* batches are r_1, $r_2,...,r_n$, correspondingly. These are unknown quantities.

If two components are purchased from the same, randomly selected batch with probability (1/*n*), the probability that both components will be of the variety 'reliable components' is given by $p_1 = \frac{1}{n}\sum_{i=1}^{n} r_i^2$.

If the two components are selected from two different, randomly selected batches, the probability that both components will be reliable is given by

$$p_2 = \frac{1}{[n(n-1)/2]}\sum_{i<j} r_i r_j = \frac{2}{n(n-1)}\sum_{i<j} r_i r_j$$

It can be shown that $p_1 \geq p_2$:

$$\frac{1}{n}\sum_{i=1}^{n} r_i^2 \geq \frac{1}{[n(n-1)/2]}\sum_{i<j} r_i r_j \qquad (17)$$

It needs to be pointed out that each batch has an equal chance, equal to 1/n, of being randomly selected.

Inequality (17) can be obtained from the classical Muirhead's inequality (Hardy et al., 1999).

Let $\{c\}$ be a non-decreasing sequence of non-negative real numbers, where $c_1 \geq c_2 \geq ... \geq c_n$ and $\{d\}$ be another non-decreasing sequence where $d_1 \geq d_2 \geq ... \geq d_n$. The sequence $\{c\}$ majorises the sequence $\{d\}$ if the following is fulfilled:

$c_1 \geq d_1$; $c_1 + c_2 \geq d_1 + d_2$; $c_1 + c_2 + ... + c_{n-1} \geq d_1 + d_2 + ... + d_{n-1}$ and $c_1 + c_2 + ... + c_{n-1} + c_n = d_1 + d_2 + ... + d_{n-1} + d_n$.

Indeed, if a sequence $\{c\}$ majorises the sequence $\{d\}$ and $r_1, r_2, ..., r_n$ are non-negative, the Muirhead inequality states that

$$\sum_{sym} r_1^{c_1} r_2^{c_2} ... r_n^{c_n} \geq \sum_{sym} r_1^{d_1} r_2^{d_2} ... r_n^{d_n} \qquad (18)$$

where the symmetric sum $\sum_{sym} r_1^{c_1} r_2^{c_2} ... r_n^{c_n}$ is obtained by adding the terms corresponding to all distinct permutations of the elements of the sequence $\{c\}$ while the symmetric sum $\sum_{sym} r_1^{d_1} r_2^{d_2} ... r_n^{d_n}$ is obtained by adding the terms corresponding to all distinct permutations of the elements of the sequence $\{d\}$.

Consider now the sequence $\{c\} = [2, 0, ..., 0]$ and the sequence $\{d\} = [1, 1, 0, ..., 0]$. Sequence $\{c\}$ majorises sequence $\{d\}$ because $2 > 1$, $2 + 0 \geq 1 + 1$ and $2 + 0 + 0 \geq 1 + 1 + 0, ...,$ $2 + 0 + 0 + ... + 0 \geq 1 + 1 + 0 + ... + 0$.

Consequently, the Muirhead inequality

$$\sum_{sym} r_1^2 r_2^0 ... r_n^0 \geq \sum_{sym} r_1^1 r_2^1 ... r_n^0 \qquad (19)$$

holds. Inequality (19) is equivalent to

$$(n-1)![r_1^2 + r_2^2 + ... + r_n^2] \geq (n-2)! \times 2\sum_{i<j} r_i r_j \quad (20)$$

Dividing both sides of inequality (20) by $n!$ gives inequality (17).

To simplify the interpretation of inequality (17) consider the special case of inequality (17), for $n = 3$:

$$(1/3)a^2 + (1/3)b^2 + (1/3)c^2 \geq (1/3)ab + (1/3)bc + (1/3)ca \quad (21)$$

where *a*,*b* and *c* are real values. Let *a*,*b* and *c* in inequality (21) stand for the fractions of reliable components in three batches (suppliers) *A*,*B* and *C*: $0 \leq a \leq 1$; $0 \leq b \leq 1$ and $0 \leq c \leq 1$. The fractions *a*,*b* and *c*, characterising the reliable components in the separate batches are unknown quantities. The left-hand side of inequality (21) can be interpreted as probability of selecting two reliable components from a randomly selected batch (supplier). Indeed, two reliable components from a randomly selected batch can be selected in three mutually exclusive ways:

(i) Batch *A* is randomly selected with probability 1/3, followed by selecting two reliable components from batch *A* (the probability of this compound event is $(1/3)a^2$);

(ii) Batch *B* is randomly selected with probability 1/3, followed by selecting two reliable components from batch *B* (the probability of this compound event is $(1/3)b^2$); and (iii) Batch *C* is randomly selected with probability 1/3, followed by selecting two reliable components from batch *C* (the probability of this compound event is $(1/3)c^2$). Since these are mutually exclusive events, the probability of their union is a sum of the probabilities of the separate events.

In a similar fashion, the right-hand side of inequality (21) can be interpreted as the probability of selecting two reliable components from two different, randomly selected batches. Indeed, two reliable components from two randomly selected batches can be selected in three mutually exclusive ways:

(i) The pair of batches *A,B* is randomly selected with probability 1/3, followed by selecting two reliable components from these batches (the probability of this compound event is $(1/3)ab$);

(ii) The pair of batches *B,C* is randomly selected with probability 1/3, followed by selecting two reliable components from these batches (the probability of this compound event is $(1/3)bc$);

(iii) The pair of batches *C,A* is randomly selected with probability 1/3, followed by selecting two reliable components from these batches (the probability of this compound event is $(1/3)ca$).

Since these are also mutually exclusive events, the probability of their union is the sum of the probabilities of the separate events.

The interpretation of inequality (21) yields a counter-intuitive result: *Irrespective of the fractions of reliable components characterising the individual batches (suppliers), selecting two components from a single batch (supplier) is always associated with a larger probability of getting two reliable components than the corresponding probability if the two components are selected from two different batches*.

The edge provided by this strategy can be significant as the next numerical example clearly demonstrates. Suppose that the fractions of reliable components (unknown to the purchaser) are equal to $a = 0.85$, $b = 0.24$ and $c = 0.57$. The probability of selecting two reliable components from a randomly selected single batch is given by:

$$p_1 = (1/3)a^2 + (1/3)b^2 + (1/3)c^2 = $$
$$= (1/3) \times 0.85^2 + (1/3) \times 0.24^2 + (1/3) \times 0.57^2 = $$
$$= 0.368$$

while the probability of selecting two reliable components from a randomly selected pair of batches is given by:

$$p_2 = (1/3)ab + (1/3)bc + (1/3)ca =$$
$$= (1/3)\times 0.85\times 0.24 + (1/3)\times 0.24\times 0.57 +$$
$$+(1/3)\times 0.57\times 0.85 = 0.275$$

As a result, $p_1 > p_2$.

It must be pointed out that selecting two components from a single, randomly selected batch will also maximise the probability that both components will be faulty.

This, however, does not mean that it is more beneficial to select the components from two different batches. Despite that selecting from two different batches decreases the probability that both selected components will be faulty, for a faulty system to be present it is not necessary both selected components to be faulty. Selecting a single faulty component is sufficient. Selecting components from two different batches decreases the probability of having a reliable system (two reliable components). This conclusion remains unchanged if the fractions of reliable components are considered to be fractions of faulty components.

Indeed, if we consider $a = 0.85$, $b = 0.24$ and $c = 0.57$, to be the fractions of faulty components in the batches, the probability of selecting two reliable components from a randomly selected batch is

$$p_1 = (1/3)[(1-a)^2 + (1-b)^2 + (1-c)^2]$$
$$= (1/3)[0.15^2 + 0.76^2 + 0.43^2] = 0.262$$

while the probability of selecting two reliable components from two randomly selected batches is

$$p_2 =$$
$$= (1/3)[(1-a)(1-b) + (1-b)(1-c) + (1-c)(1-a)]$$
$$= (1/3)[0.15\times 0.76 + 0.76\times 0.43 + 0.43\times 0.15] = 0.168$$

Again, $p_1 > p_2$.

No matter what a, b and c denote (percentage of reliable components or percentage of faulty components) the probability that both components will be reliable is always maximised by selecting both components from the same randomly selected batch. With this, the probability of a reliable system is also maximised.

In a similar fashion, by using the Muirhead's inequality, inequality (13) can be generalised for more than two selected components. (The reasoning is very similar to the reasoning in deriving inequality (13) and will not be repeated).

Thus, for three selected components the inequality:

$$(1/3)a^3 + (1/3)b^3 + (1/3)c^3 \geq$$
$$\geq (1/6)a^2b + (1/6)a^2c + (1/6)b^2a + \quad (22)$$
$$+(1/6)b^2c + (1/6)c^2a + (1/6)c^2b$$

follows directly from the Muirhead's inequality (18).

The right-hand side of inequality (22) is the probability of selecting three reliable components from two different, randomly selected batches.

The inequality

$$(1/3)a^3 + (1/3)b^3 + (1/3)c^3 \geq abc \quad (23)$$

can also be obtained from the Muirhead's inequality (18). The right-hand side of this inequality is the probability of selecting three reliable components from three different batches.

Inequalities (22) and (23) can be interpreted as follows. *Irrespective of the fractions of reliable components characterising the individual batches, selecting three components from a single, randomly selected batch is always associated with a larger probability of selecting three reliable components than the probability of selecting three reliable components from different batches*.

For four selected components, the inequality:

$$(1/3)a^4 + (1/3)b^4 + (1/3)c^4 \geq$$
$$\geq (1/6)a^3b + (1/6)a^3c + (1/6)b^3a + \quad (24)$$
$$+(1/6)b^3c + (1/6)c^3a + (1/6)c^3b$$

can be obtained from the Muirhead's inequality (18). The right-hand side of inequality (24) is the probability of selecting four reliable components from two batches if three components are selected from one of the batches and one component from the other batch.

The inequality:

$$(1/3)a^4 + (1/3)b^4 + (1/3)c^4 \geq$$
$$\geq (1/3)a^2b^2 + (1/3)b^2c^2 + (1/3)a^2c^2 \quad (25)$$

can also be obtained from the Muirhead's inequality (18). The right-hand side of inequality (25) is the probability of selecting four reliable components from two randomly selected batches if two components are selected from one batch and two components from the other batch.

Finally, the inequality:

$$(1/3)a^4 + (1/3)b^4 + (1/3)c^4 \geq$$
$$\geq (1/3)a^2bc + (1/3)b^2ac + (1/3)c^2ab \quad (26)$$

can be obtained from the Muirhead's inequality. The righthand side of inequality (26) is the probability of selecting four reliable components from three randomly selected batches if two components are selected from one of the batches and two components from each of the other two batches.

Inequalities (24)-(26) can be interpreted as follows. *Irrespective of the fractions of reliable components characterising the individual batches, selecting four components from a single batch is always associated with a larger probability of selecting four reliable components than the probability of selecting four reliable components from different batches*.

The generalisation of these results for different number of components and different number of batches (suppliers) leads to the following counter-intuitive result related to suppliers delivering the same type of products and characterised by unknown fractions of the reliable products they deliver. *Irrespective of the fractions of reliable products characterisng the individual suppliers, purchasing all products from a single, randomly selected supplier, maximises the probability that all purchased products will be reliable.*

CONCLUSIONS

1. New results related to the reliability of common systems with dual redundancies at a component level have been obtained by a physical interpretation of algebraic inequalities. The knowledge derived from the interpretation of the inequalities can be used for increasing the system reliability.

2. For systems with interchangeable redundant components, the system reliability corresponding to a symmetric arrangement of the redundant components is always inferior to the system reliability corresponding to an asymmetric arrangement of the redundant components. This result holds irrespective of the probabilities of failure characterising the different types of components.

3. For a system with components logically arranged in series, with interchangeable redundancies at a component level, the system reliability is maximised by arranging the main components in ascending order of their probabilities of failure while the redundant components are arranged in descending order of their probabilities of failure.

4. The interpretation of an algebraic inequality led to a counter-intuitive result related to suppliers delivering the same type of product and characterised by unknown fractions of the delivered reliable producrs. Irrespective of the fractions of reliable products characterisng the individual suppliers, purchasing all products from a single, randomly selected supplier maximises the probability that all purchased products will be reliable.

APPENDIX. Proof of inequality (15)

Inequality (15) can be proved by induction, by proving first the case for two sections ($n = 2$):

$$(1 - a_1^m)(1 - a_2^m) \leq$$
$$(1 - a_1^{m-t} a_2^t)(1 - a_2^{m-t} a_1^t) \quad (A1)$$

$0 \leq a_1 \leq 1$; $0 \leq a_2 \leq 1$; $a_1 \leq a_2$, $1 \leq t \leq m$.

Proving this inequality is equivalent to proving the equivalent inequality

$$1-a_1^m - a_2^m + a_1^m a_2^m \le$$
$$1-a_1^{m-t}a_2^t - a_2^{m-t}a_1^t + a_1^m a_2^m \quad \text{(A2)}$$

which, in turn, can be proved by proving the equivalent inequality

$$a_1^m + a_2^m - a_1^{m-t}a_2^t - a_1^t a_2^{m-t} \ge 0 \quad \text{(A3)}$$

Inequality (A3) however, is true because $a_1^m + a_2^m - a_1^{m-t}a_2^t - a_1^t a_2^{m-t} = (a_2^t - a_1^t)(a_2^{m-t} - a_1^{m-t})$ is non-negative.

Inequality (15) can now be proved by induction. For $n=2$, inequality (15) coincides with inequality (A2) which has been shown to be true.

The probabilities of failure of the components can always be ordered in ascending ($a_1 \le a_2 \le ... \le a_k \le a_{k+1}$) order. Let us assume that inequality (11) is true for $n=k$ (induction hypothesis):

$$(1-a_1^m)(1-a_2^m)...(1-a_k^m) \le$$
$$(1-a_1^{m-t}a_2^t)(1-a_1^{m-t}a_3^t)...(1-a_k^{m-t}a_1^t) \quad \text{(A4)}$$

We will show that the inequality is also valid for $n=k+1$.

Multiplying both sides of inequality (A4) by $(1-a_{k+1}^m)$ gives the inequality

$$(1-a_1^m)(1-a_2^m)...(1-a_k^m)(1-a_{k+1}^m) \le$$
$$(1-a_1^{m-t}a_2^t)(1-a_2^{m-t}a_3^t)(1-a_k^{m-t}a_1^t)(1-a_{k+1}^m) \quad \text{(A5)}$$

If it can be shown that

$$(1-a_k^{m-t}a_1^t)(1-a_{k+1}^m) \le$$
$$(1-a_k^{m-t}a_{k+1}^t)(1-a_{k+1}^{m-t}a_1^t) \quad \text{(A6)}$$

This means that replacing the expression $(1-a_k^{m-t}a_1^t)(1-a_{k+1}^m)$ in the right-hand side of inequality (A5) by the larger expression $(1-a_k^{m-t}a_{k+1}^t)(1-a_{k+1}^{m-t}a_1^t)$, will only strengthen inequality (A5).

Consequently, to prove inequality (A6), the equivalent inequality:

$$1-a_{k+1}^m - a_k^{m-t}a_1^t + a_1^t a_k^{m-t} a_{k+1}^m \le$$
$$1-a_{k+1}^{m-t}a_1^t - a_k^{m-t}a_{k+1}^t + a_1^t a_k^{m-t} a_{k+1}^m \quad \text{(A7)}$$

must be proved, which is obtained from expanding the left- and right-hand side of (A6). Proving inequality (A7) is equivalent to proving

$$a_{k+1}^m + a_k^{m-t}a_1^t - a_{k+1}^{m-t}a_1^t - a_k^{m-t}a_{k+1}^t \ge 0 \quad \text{(A8)}$$

The left-hand side of (A8) can be factorised as:

$$a_{k+1}^m + a_k^{m-t}a_1^t - a_{k+1}^{m-t}a_1^t - a_k^{m-t}a_{k+1}^t =$$
$$= (a_{k+1}^t - a_1^t)(a_{k+1}^{m-t} - a_k^{m-t}) \quad \text{(A9)}$$

and because $a_1 \le a_2 \le ... \le a_k \le a_{k+1}$, $(a_{k+1}^t - a_1^t)(a_{k+1}^{m-t} - a_k^{m-t}) \ge 0$. The case $n=k+1$ has been proved. Because inequality (15) is true for $n=2$, the inequality is also valid for $n=3$, $n=4$ and for any other $n \ge 2$.

The same argument is valid if the probabilities of failure were arranged in descending ($a_1 \ge a_2 \ge ... \ge a_k \ge a_{k+1}$) order.

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